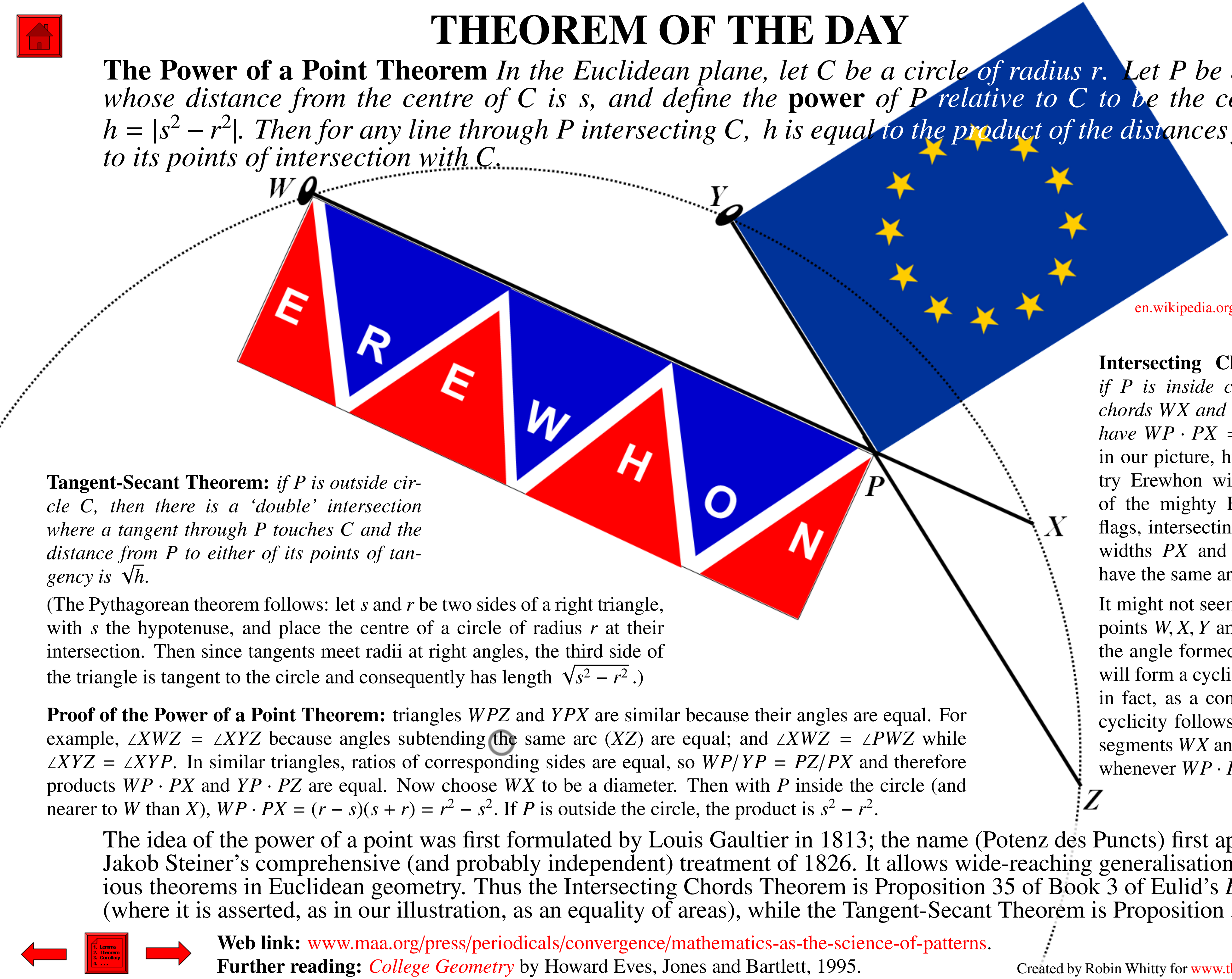
THEOREM OF THE DAY

The Power of a Point Theorem *In the Euclidean plane, let C be a circle of radius r. Let P be a point whose distance from the centre of C is s, and define the* **power** *of P relative to C to be the constant* $h = |s^2 - r^2|$. *Then for any line through P intersecting C, h is equal to the product of the distances from P to its points of intersection with C.*



Tangent-Secant Theorem: *if P is outside circle C, then there is a 'double' intersection where a tangent through P touches C and the distance from P to either of its points of tangency is* $\sqrt{h}$.

(The Pythagorean theorem follows: let s and r be two sides of a right triangle, with s the hypotenuse, and place the centre of a circle of radius r at their intersection. Then since tangents meet radii at right angles, the third side of the triangle is tangent to the circle and consequently has length $\sqrt{s^2 - r^2}$.)

Intersecting Chords Theorem: *if P is inside circle C, then for chords WX and YZ through P, we have* $WP \cdot PX = YP \cdot PZ$. Thus, in our picture, humble little country Erewhon wishes it were part of the mighty EU; at least their flags, intersecting at P and having widths PX and PZ, respectively, have the same area...

It might not seem obvious that our points W, X, Y and Z, regardless of the angle formed by the flagstaffs, will form a cyclic quadrilateral but in fact, as a converse result, concyclicity follows for arbitrary line segments WX and YZ meeting at P whenever $WP \cdot PX = YP \cdot PZ$.

Proof of the Power of a Point Theorem: triangles WPZ and YPX are similar because their angles are equal. For example, $\angle XWZ = \angle XYZ$ because angles subtending the same arc (XZ) are equal; and $\angle XWZ = \angle PWZ$ while $\angle XYZ = \angle XYP$. In similar triangles, ratios of corresponding sides are equal, so $WP/YP = PZ/PX$ and therefore products $WP \cdot PX$ and $YP \cdot PZ$ are equal. Now choose WX to be a diameter. Then with P inside the circle (and nearer to W than X), $WP \cdot PX = (r - s)(s + r) = r^2 - s^2$. If P is outside the circle, the product is $s^2 - r^2$.

The idea of the power of a point was first formulated by Louis Gaultier in 1813; the name (Potenz des Puncts) first appears in Jakob Steiner's comprehensive (and probably independent) treatment of 1826. It allows wide-reaching generalisations of various theorems in Euclidean geometry. Thus the Intersecting Chords Theorem is Proposition 35 of Book 3 of Eulid's *Elements* (where it is asserted, as in our illustration, as an equality of areas), while the Tangent-Secant Theorem is Proposition 36.

Web link: www.maa.org/press/periodicals/convergence/mathematics-as-the-science-of-patterns.

Further reading: *College Geometry* by Howard Eves, Jones and Bartlett, 1995.