THEOREM OF THE DAY

Euler's Identity *With τ and e the real constants*

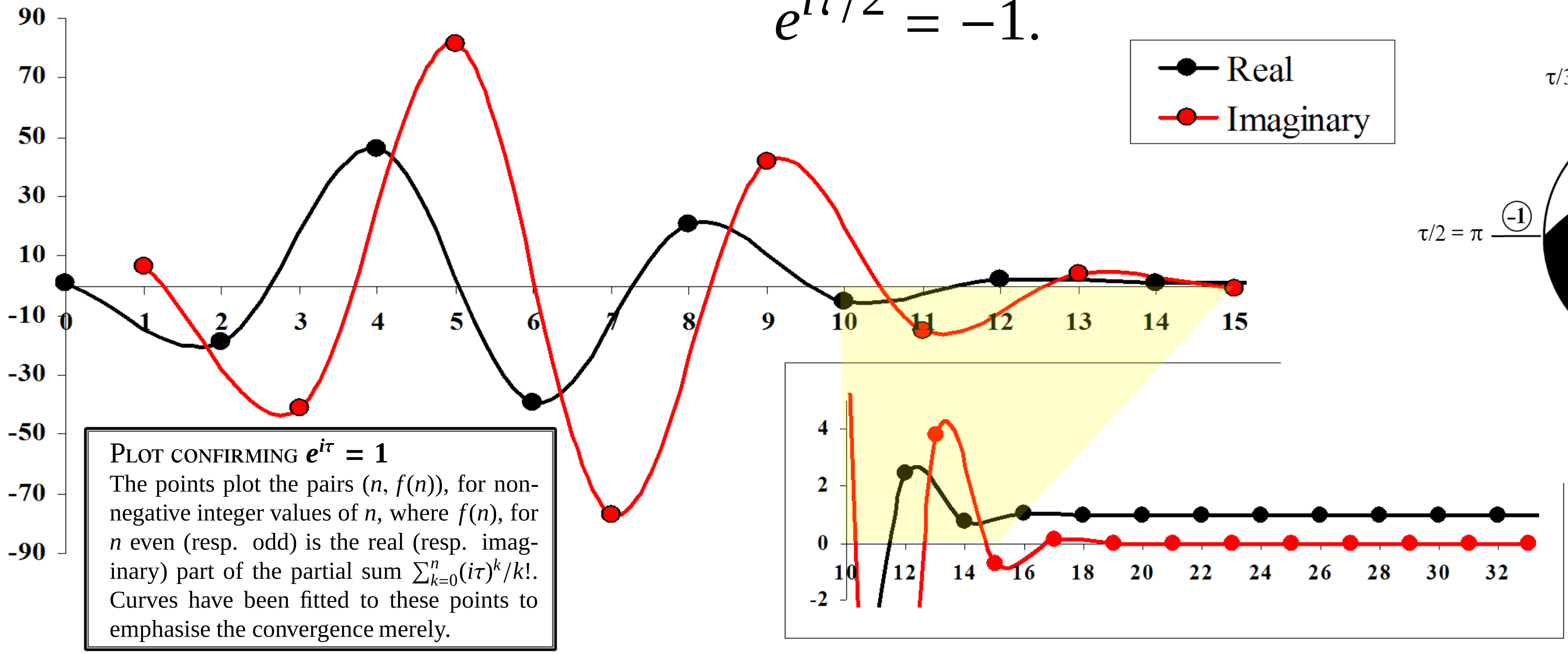
$$\tau = 6.2831853071\ 7958647692\ 5286766559\ 0057683943\ 3879875021\ 1641949889\ 1846156328\ 1257241799\ 7256069650\ 6842341359\ldots$$

and

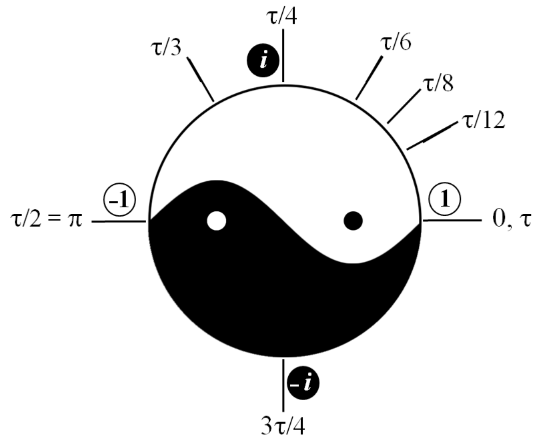
$$e = 2.7182818284\ 5904523536\ 0287471352\ 6624977572\ 4709369995\ 9574966967\ 6277240766\ 3035354759\ 4571382178\ 5251664274\ldots$$

(the first 100 places of decimal being given), and i the imaginary constant satisfying $i^2 = -1$, we have

$$e^{i\tau/2} = -1.$$



Plot confirming $e^{i\tau} = 1$
The points plot the pairs $(n, f(n))$, for non-negative integer values of n, where $f(n)$, for n even (resp. odd) is the real (resp. imaginary) part of the partial sum $\sum_{k=0}^{n}(i\tau)^k/k!$. Curves have been fitted to these points to emphasise the convergence merely.



Squaring both sides of $e^{i\tau/2} = -1$ gives $e^{i\tau} = 1$, encoding the defining fact that τ radians measures one full circle. The calculation can be confirmed explicitly using the evaluation of e^z, for any complex number z, as an infinite sum: $e^z = 1 + z + z^2/2! + z^3/3! + z^4/4! + \ldots$. Setting $z = i\tau$, the even powers of i alternate between 1 and -1, while the odd powers alternate between i and $-i$, so we get two separate sums, one with i's (the imaginary part) and one without (the real part). Both converge rapidly as shown in the two plots above: the real part to 1, the imaginary to 0. In the *limit* equality is attained, $e^{i\tau} = 1 + 0 \times i$, whence $e^{\tau i} = 1$. The value of $e^{i\tau/2}$ may be confirmed in the same way.

Leonhard Euler's 1748 *Introductio* presents the general circle identity $e^{i\theta} = \cos\theta + i\sin\theta$, with $\theta = \tau/2$ radians (half a turn) giving the iconic evaluation of $e^{i\tau/2}$. Although better known in the form $e^{i\pi} + 1 = 0$, $\pi = \tau/2$, the half circle angle $\tau/2$ is essential. Thus $3i\tau/2, 5i\tau/2, \ldots$, also exponentiate to -1, $\tau/2$ being distinguished as the **principal value**. A quarter turn $\tau/4$ gives $e^{i\tau/4} = i$, whence the remarkable fact that $i^i = \left(e^{i\tau/4}\right)^i = e^{i^2\tau/4} = e^{-\tau/4}$, a real number. And in general $e^{i\tau/n} = \sqrt[n]{1}$, an n-th root of unity.

Web link: fermatslasttheorem.blogspot.com/2006/02/eulers-identity.html. More on i^i: www.walkingrandomly.com/?p=294.

Further reading: *Dr Euler's Fabulous Formula: Cures Many Mathematical Ills,* by Paul J. Nahin, Princeton University Press, 2006