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The triangle-halving deltoid envelope

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In an article in 1972 [1] (and see [2] for further developments), Dunn and Pretty determined the collection of all straight lines bisecting the area of a triangle as a certain 'deltoid' envelope. More specifically, they observed that the general problem could be reduced to bisecting a right isosceles triangle at the origin, and that in this triangle the bisecting lines were the tangents to a deltoid curve traced by three hyperbolae. The hyperbolae in question were those whose asymptotes were the pairs of edges of the triangle (we are only concerned with the positive, or right-facing branches of the hyperbolae, suitably rotated).

Figure 1 shows Dunn and Pretty's deltoid envelope for an arbitrarily specified triangle ABC (whose edges have been produced the more easily to picture them as asymptotes). There are six area-bisecting lines which are easy to identify, plotted as grey lines. These are the three triangle medians, joining vertices to midpoints of opposite sides; and the three lines parallel to the triangle edges and dividing the altitudes in the ratio $1/\sqrt{2} : 1 - 1/\sqrt{2}$. We observe that these six lines are, indeed, tangent to the plotted hyperbolae.

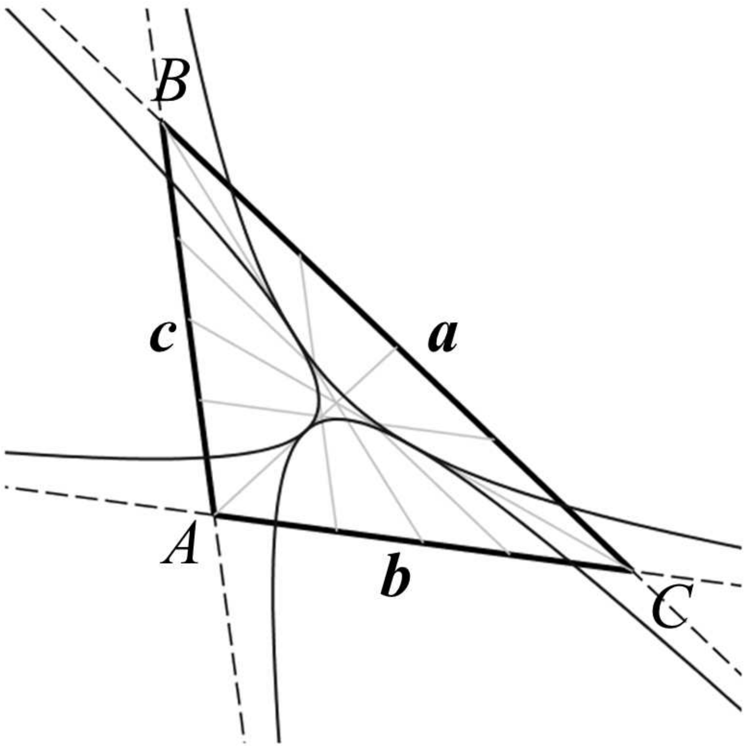

Fig. 1

The purpose of this note is to record how we plotted the hyperbolae in Figure 1. There is a grown-up way we could have done it, which is to carry out Dunn and Pretty's affine transformation to a triangle placed at the origin, and then apply the reverse transformation to their hyperbolae. We would like to see some *M500* reader crack this in a future issue! Still, it is a useful exercise to acquire the necessary ingredients from scratch, so to speak, for a given triangle. And this is what we will do here.

A hyperbola may be specified in terms of various subsets of its parameters: foci, vertices, centre, eccentricity, asymptotes, etc. Our aim will be to give values to those parameters necessary to write down each hyperbola in parametric form thus:

$$\begin{aligned} x &= x_0 + p\cosh(t)\cos\theta - q\sinh(t)\sin\theta, \\ y &= y_0 + p\cosh(t)\sin\theta + q\sinh(t)\cos\theta, \end{aligned}$$

with t ranging over a half-circle, $-\tau/2 \le t \le \tau/2$. The angle θ gives the rotation from the hyperbola symmetric about the horizontal axis. This is more explicit if we write these equations in terms of a rotation matrix:

$$(x, y) = (x_0, y_0) + (p\cosh(t), q\sinh(t)) \times \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}. \tag{1}$$

We will start by finding θ, the angle of rotation. Of course we need this angle to correctly orient our hyperbola. In addition it includes the angle between one asymptote of the hyperbola and the midline between the asymptotes, and the tangent of this latter angle is q/p. So θ is doubly useful to us!

From now on we will base our calculations on the triangle ABC in Figure 1 and restrict our attention to the hyperbola whose asymptotes are edges $\mathbf{a}$ and $\mathbf{c}$. We will regard these edges as free vectors, oriented clockwise around the triangle, so that $\mathbf{a} = -B + C$, and so on, treating A, B and C as position vectors. The two other hyperbolae may be obtained in an identical manner.

We start with a right-facing half-hyperbola symmetrical about the horizontal axis. We rotate by a quarter circle clockwise, $-\tau/4$: the hyperbola is now downwards facing, about the vertical axis. We rotate back, anticlockwise, by the angle between the vertical axis and triangle edge $\mathbf{c}$. This is obtained from the dot product of the unit vectors in the directions of $\mathbf{c}$ and of the vertical axis as:

$$\cos^{-1}\frac{\mathbf{c}\cdot(0,1)}{|\mathbf{c}|}. \tag{2}$$

Finally we must rotate anticlockwise by an angle, call it ϕ, which is half the angle between $\mathbf{c}$ and $-\mathbf{a}$. Now $\cos(2\phi) = -\mathbf{c}\cdot\mathbf{a}/(|\mathbf{c}||\mathbf{a}|)$. So, using the appropriate half-angle formula,

$$\tan(2\phi/2) = \sqrt{\frac{1-\cos(2\phi)}{1+\cos(2\phi)}} = \sqrt{\frac{1+\mathbf{c}\cdot\mathbf{a}/(|\mathbf{c}||\mathbf{a}|)}{1-\mathbf{c}\cdot\mathbf{a}/(|\mathbf{c}||\mathbf{a}|)}},$$

giving

$$\phi = \tan^{-1}\sqrt{X_{ac}} \quad \text{where} \quad X_{ac} = \frac{|\mathbf{a}||\mathbf{c}| + \mathbf{a}\cdot\mathbf{c}}{|\mathbf{a}||\mathbf{c}| - \mathbf{a}\cdot\mathbf{c}}. \tag{3}$$

To sum up, the angle of rotation of the hyperbola whose asymptotes are $\mathbf{c}$ and $\mathbf{a}$ may be written

$$\theta_{ac} = -\frac{\tau}{4} + \cos^{-1}\frac{\mathbf{c}\cdot(0,1)}{|\mathbf{c}|} + \tan^{-1}\sqrt{X_{ac}}. \tag{4}$$

Moreover, as we said earlier with reference to equation (1), $\tan\phi = q/p$, so we can record:

$$q = p\sqrt{X_{ac}}. \tag{5}$$

This means that there effectively remains just one unknown on the right-hand side of equation (1) and that is p. To find p we return to this equation, using the identity (5) and some shorthand, $X = x - x_0, Y = y - y_0, \mathbf{C} = \cos\theta_{ac}, \mathbf{S} = \sin\theta_{ac}$:

$$(X, Y) = (\cosh(t), \sinh(t)) \times \begin{pmatrix} p & 0 \\ 0 & p\sqrt{X_{ac}} \end{pmatrix} \times \begin{pmatrix} \mathbf{C} & \mathbf{S} \\ -\mathbf{S} & \mathbf{C} \end{pmatrix}. \tag{6}$$

Inverting the matrices:

$$(X, Y)\begin{pmatrix} \mathbf{C} & -\mathbf{S} \\ \mathbf{S} & \mathbf{C} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1/\sqrt{X_{ac}} \end{pmatrix} = p\,(\cosh(t), \sinh(t))\,.$$

Now apply the identity $\cosh^2(t) - \sinh^2(t) = 1$ and equate coordinates:

$$(X\mathbf{C} + Y\mathbf{S})^2 - \frac{(Y\mathbf{C} - X\mathbf{S})^2}{X_{ac}} = p^2$$

or as a vector length:[1]

$$p = \left| \begin{pmatrix} 1 & 0 \\ 0 & 1/\sqrt{-X_{ac}} \end{pmatrix} \begin{pmatrix} \mathbf{C} & \mathbf{S} \\ -\mathbf{S} & \mathbf{C} \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} \right|. \tag{7}$$

This is the distance between the centre of the hyperbola and its vertex. The centre, which is (x_0, y_0) in equation (1), is the point of intersection of the asymptotes. This is point B for our present purposes. To evaluate p we need to locate a point (x, y) lying on the hyperbola in order to give values to $X = x - x_0$ and $Y = y - y_0$. We will take that point on the bisecting line parallel to edge $\mathbf{b}$ which is tangent to the hyperbola. This point also lies on the median line from vertex B.[2] It therefore divides the median line in the ratio $1/\sqrt{2} : 1 - 1/\sqrt{2}$. The median line, as a free vector may be written as $1/2(\mathbf{a} - \mathbf{c})$. So the required point, lying on the hyperbola asymptotic to $\mathbf{a}$ and $\mathbf{c}$, is $B + 1/(2\sqrt{2})(\mathbf{a} - \mathbf{c})$. And this gives us the final ingredients for our plot:

$$(X, Y) = \frac{1}{2\sqrt{2}}(\mathbf{a} - \mathbf{c}).$$

References

[1] Dunn, J.A. and Pretty, J.E., "Halving a triangle", *Math. Gaz.* Vol. 56, No. 396, 1972, pp. 105-108.

[2] Berele, A. and Catoiu, S., "Bisecting the perimeter of a triangle", *Mathematics Magazine*, Vol. 91, No. 2, 2018, pp. 121–133.

[1] The square root will be imaginary, so this is perhaps a notational convenience. Nevertheless, it seems there should be some direct link from equation (6) to the vector length. Another puzzle for *M500* readers!

[2] This may presumably be confirmed retrospectively by differentiating the function we are in the process of specifying! It seems it should be obvious geometrically but this eludes us. Yet another *M500* reader challenge.