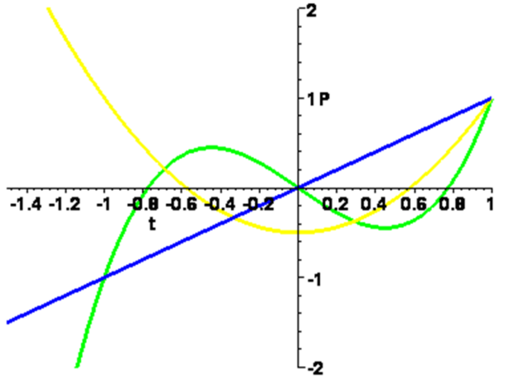
THEOREM OF THE DAY

The Delsarte-Goethals-Seidel Theorem *For some integer $d \geq 0$, let $f(t) = \sum_{k=0}^{d} c_k G_k^n(t)$ be a summation over an ortho-normal set of $d+1$ Gegenbauer polynomials, with $c_k \geq 0$, such that, for some real $C > 0$,*

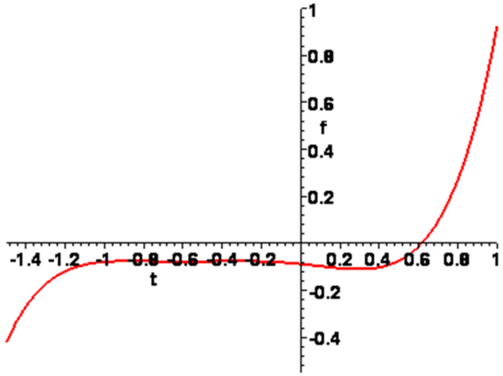
(i) $f(t) \leq -C$, in the range $[-1, 1/2]$; and

(ii) $f(1) \leq 1 - C$.

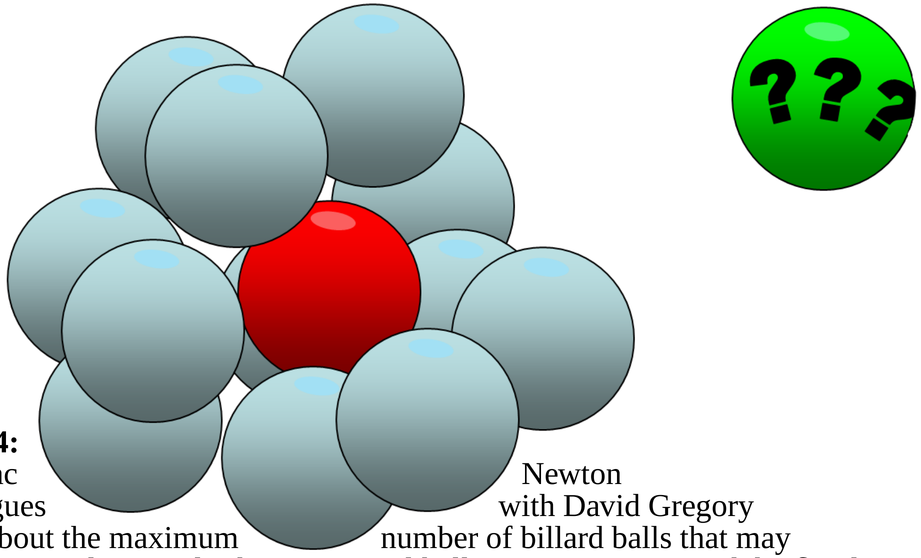
Then $\kappa(n)$, the kissing number *in dimension n, obeys $\kappa(n) \leq 1/C$.*



$P_k(x)$, for $k = 1$ (blue), 2 (yellow) and 3 (green)



$f(t) = 0.21P_1(x) + 0.28P_2(x) + 0.25P_3(x) + 0.14P_4(x) + 0.04P_5(x)$



1694: Isaac Newton argues with David Gregory about the maximum number of billard balls that may simultaneously 'kiss' a central ball; is it 12 or can a 13th be fitted?
1953: the latter is finally ruled out by Schütte and van der Waerden.

In dimension $n = 3$, normalised Gegenbauer polynomials are just the familiar Legendre polynomials: $P_0(t) = 1$, $P_1(t) = t$, $P_2(t) = (3t^2 - 1)/2$, $P_3(t) = t(5t^2 - 3)/2$, $P_4(t) = (35t^4 - 30t^2 + 3)/8$, $P_5(t) = t(63t^4 - 70t^3 + 15t)/8, \ldots$, see top right, above. Now, bottom right, the summation, $f(t)$, satisfies the theorem with $C \approx 0.075$, giving $\kappa(3) < 13.33$. In fact, no summation achieves $1/C < 13$, but this subtle approach comes close to answering Newton vs Gregory with a single, simple calculation. In higher dimensions it sometimes applies even more spectacularly.

Philippe Delsarte's innovative linear programming approach to problems in coding and design theory is seen in action, in this 1977 theorem, revolutionising the study of a classical problem. This led eventually to the proof by Oleg Musin, in 2003, that $\kappa(4) = 24$. Above that, the bound of Delsarte, Jean-Marie Goethals and Johan Jacob Seidel is the only effective one known; for deep geometrical reasons it solves the problem exactly in dimensions 8 and 24.

Web link: plus.maths.org/issue23/features/kissing, with more historical background at www.ams.org/notices/200408/comm-cass.pdf.
The theorem as stated here is from: arxiv.org/abs/math/0501493.
Further reading: *From Error-Correcting Codes through Sphere Packings to Simple Groups* by T. M. Thompson, The MAA, 2004.