THEOREM OF THE DAY

De Moivre's Theorem *Let θ be an angle and n a positive integer. Then*

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta.$$

F18 {=MMULT(ROOT3;E18:E19)}

Rotation by Tau/n (anticlockwise)
(Tau = 2*Pi)

=Cos(2Pi/n)	=-Sin(2Pi/n)
=Sin(2Pi/n)	=Cos(2Pi/n)

ROOT3 (n=3)

-0.5000	-0.866
0.8660	-0.500

ROOT5 (n=5)

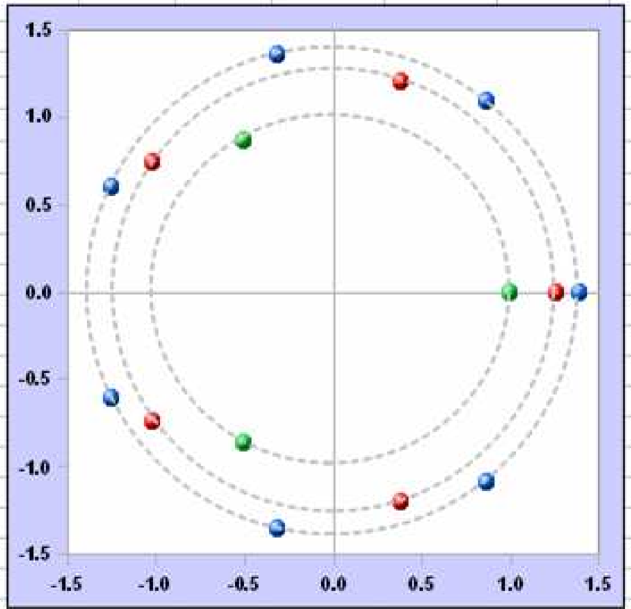
0.3090	-0.951
0.9511	0.309

ROOT7 (n=7)

0.6235	-0.782
0.7818	0.623

nth roots of

1	real	1.000	-0.500	-0.500	1.000				
n=3	complex	0.000	0.866	-0.866	0.000				
Pi	real	1.257	0.389	-1.017	-1.017	0.389	1.257		
n=5	complex	0.000	1.196	0.739	-0.739	-1.196	0.000		
10	real	1.389	0.866	-0.309	-1.252	-1.252	-0.309	0.866	1.389
n=7	complex	0.000	1.086	1.355	0.603	-0.603	-1.355	-1.086	0.000



Among other things, De Moivre's theorem is perfect for finding n-th roots of real or complex numbers, as shown in the above screenshot from OpenOffice Calc. In the plot we think of the horizontal axis as recording the real part and the vertical axis the imaginary part of the complex number $z = a + ib$. This number may be written as $z = r(\cos\theta + i\sin\theta)$ where $r = \sqrt{a^2+b^2}$ and θ is the angle made with the horizontal axis by the line from $(0,0)$ to (a,b): $\cos\theta = a/r$. On our horizontal axis three real numbers are plotted: $\sqrt[3]{1} = 1$, $\sqrt[5]{\tau/2} \approx 1.257$ ($\tau = 2\pi$) and $\sqrt[7]{10} \approx 1.389$. But there are complex n-th roots too! For instance, the complex number z will be cube root of 1 if $1 = z^3 = r^3(\cos\theta + i\sin\theta)^3 = r^3(\cos 3\theta + i\sin 3\theta)$. This will happen whenever 3θ is a multiple of τ because $\cos\tau + i\sin\tau = 1 + i\times 0 = 1$. So in our plot we have the point $(1,0)$ and this point rotated by $\tau/3$ and by $2\tau/3$ to give the three cube roots of 1. A nice way to rotate a point (x,y) by an angle τ/n is to multiply the point by the appropriate *rotation matrix* as shown above left. Note that $n = 2$ gives a rotation of the real square root of a number x by $\tau/2$, or 180°, to give $-x$.

Abraham de Moivre discovered a version of his formula in 1707 and proposed the more usual version in 1722. Leonhard Euler's famous 1749 identity $e^{i\theta} = \cos\theta + i\sin\theta$ generalised it to real number exponents.

Web link: www.cimt.org.uk/projects/mepres/alevel/alevel.htm, Ch. 3 of Further Pure Mathematics. Historical background: see note 46 on page. 118 of projecteuclid.org/euclid.ss/1185975640.

Further reading: *Trigonometric Delights* by Eil Maor, Princeton University Press, 2002