Solutions to question set for Chapter 9

1. Let $G = (V, E)$ be a graph. A subset $C \subseteq E$ is called an **edge cover** if every vertex has at least one endpoint in C. This is a dual to the idea of a vertex cover whose minimum value is $\tau(G)$; denote by $\varepsilon(G)$ the size of a minimum edge cover for G.

   (a) Find an example of a graph G for which $\varepsilon(G) > \tau(G)$.

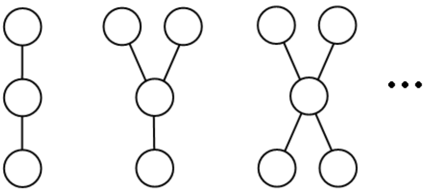
   (b) Find an example of a graph G for which $\tau(G) > \varepsilon(G)$.

   (c) Prove that if G is a cycle then $\varepsilon(G) = \tau(G)$.

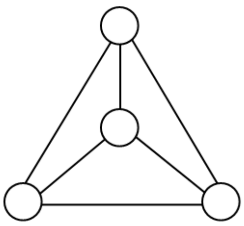
   (d) Prove that, for any G, we have $\tau(G) \leq 2\varepsilon(G)$.

SOLUTION:

(a) The extreme examples of graphs having $\varepsilon(G) > \tau(G)$ are 'star graphs' as shown below. For these graphs $\tau(G) = 1$ since the central vertex covers all edges, but $\varepsilon(G) = |E|$ since every vertex of degree 1 must be covered by a separate edge. The star graph on 3 vertices, below left, is the smallest example of a connected graph having $\varepsilon(G) > \tau(G)$.



(b) If G has a perfect matching then this minimises $\varepsilon(G)$ since the matching provides an edge cover in which each edge covers 2 vertices. So $\varepsilon(G) = \nu(G)$ for such a graph. Now if $\tau(G) > \nu(G)$ then this will be an example of a graph with $\tau(G) > \varepsilon(G)$. We have met plenty of these, e.g. the complete graph on 4 vertices:



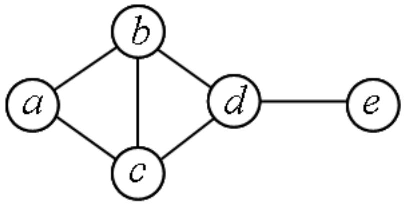
(c) Let G be a cycle of length n, denoted by C_n. If n is even then we know that $\tau(G) = \nu(G)$. And G has a perfect matching, so $\varepsilon(G) = \nu(G)$, as observed in part (b). So it remains to prove that $\varepsilon(G) = \tau(G)$ for odd n.

Suppose the vertex set of G is $V = \{x_1, x_2, \ldots, x_{2k+1}\}$, for some k, $k \geq 1$ and the edge set of G is $E = \{x_1x_2,\ x_2x_3, \ldots, x_{2k}x_{2k+1}, x_{2k+1}x_1\}$. Now $U = \{x_1, x_3, \ldots, x_{2k+1}\}$ is a vertex cover of G of size $k + 1$. Any smaller subset of V must exclude 2 consecutive vertices of the cycle which is impossible since their common adjacent edge will not be covered. So $\tau(G) = k + 1$. Similarly $F = \{x_1x_2, x_3x_4, \ldots x_{2k+1}x_1\}$ is an edge cover for G. Any smaller subset of E must exclude 2 consecutive edges of the cycle which is impossible since their common incident vertex will not be covered. So $\varepsilon(G) = k + 1$ and we have $\varepsilon(G) = \tau(G)$, as required.

(d) Suppose F is a minimum edge cover of G with $|F| = \varepsilon(G)$. Let U be the set of all vertices incident with edges of F. Then $|U| \leq 2|F|$ since each edge of F can contribute at most 2 vertices to U. And U is a vertex cover for G. For if $e = xy \in E$ is not covered by U then $x \notin U$. But then by definition of U, x belongs to no edge of F which means that x is not covered by F, contradicting the fact that F is an edge cover. So U is a vertex cover of size $2|F| = 2\varepsilon(G)$ and $\tau(G) \le 2\varepsilon(G)$, as required.

2. Suppose we want a minimum cover of the graph on the right. The following **linear programme** will achieve this:

$$\begin{array}{rl} \textbf{minimise:} & a+b+c+d+e \\ \textbf{subject to:} & \begin{array}{ccccc} a & + & b & \ge & 1 \\ a & + & c & \ge & 1 \\ b & + & c & \ge & 1 \\ b & + & d & \ge & 1 \\ c & + & d & \ge & 1 \\ d & + & e & \ge & 1 \end{array} \\ & a,b,c,d,e \ge 0, \text{integer} \end{array}$$



(a) Find a minimising solution to the inequalities on the left and explain how this corresponds to a minimum cover for the graph.

(b) What will happen if the requirement that a, b, c, d, e are integer is dropped?

SOLUTION:

(a) A solution to the inequalities which minimises $a + b + c + d + e$ is $b = c = e = 1$ and $a = d = 0$. We see that $\{b, c, e\}$ is a vertex cover for the graph and is minimum.

(b) If it is not required that a, b, c, d and e are integers then we have a solution to the inequalities $a = b = c = d = e = 1/2$ which gives $a+b+c+d+e = 5/2$. We can no longer interpret these non-integer values for $a, b, \ldots$ in terms of vertex covers so this smaller solution is a problem unless we can prevent non-integer solutions from being chosen.

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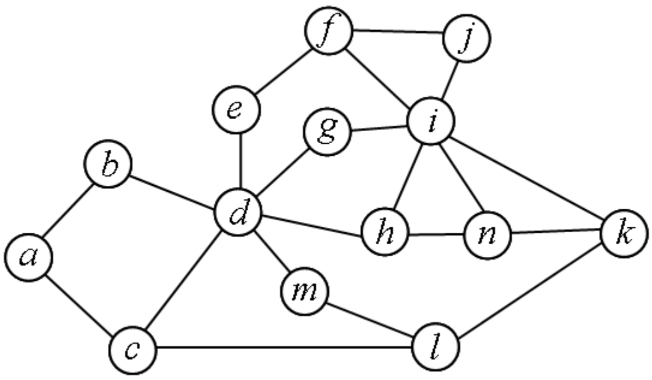
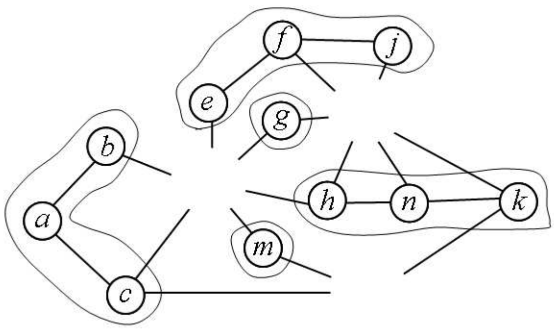
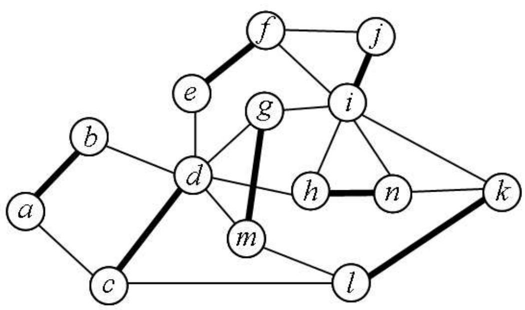
3. Suppose G is a graph and U is a subset of the vertices of G. By the *deletion* of U from G we mean removing from G all the vertices in G together with any edges which are incident with vertices in U. Denote by odd(U) the number of connected components which remain which have an odd number of vertices, when you delete U from G. One of the most celebrated theorems in graph theory[1]says that G has a perfect matching if and only if, for every subset U of vertices, odd(U) $\le |U|$.
Use the theorem to prove that the graph above-right has no perfect matching. What if an edge is added joining g to m?



SOLUTION: By taking the set $U = \{d, i, l\}$ we find that odd(U) $= 5 > |U|$ so by Tutte's theorem there can be no perfect matching. This is illustrated below-left. If edge gm is added then we *do* have a perfect matching: one such matching is illustrated below-right.

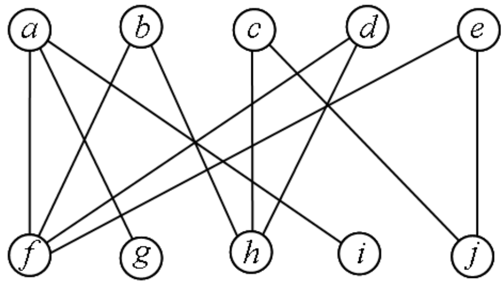




[1]Due to Bill Tutte, 1954

4. Run Kőnig's Bipartite Matching Algorithm for **two** iterations on the graph G below, showing clearly:
   (a) the steps by which a maximal alternating forest F is constructed during each iteration;
   (b) an augmenting path or collection of augmenting paths lying in F;
   (c) the augmented matching in G obtained by using the result of (b).

   Finally perform a **third** iteration and show that it: (1) confirms that a maximum matching has been found; and (2) identifies a minimum cover for G.

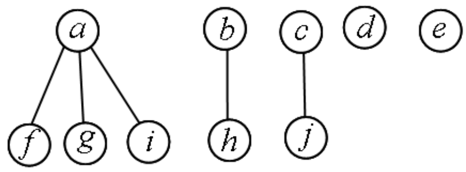


SOLUTION: In the diagrams below, heavy bold edges represent matching edges.

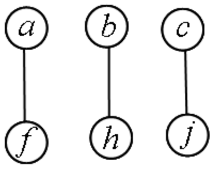
The **first** iteration can be shown diagrammatically as follows:

Step 1

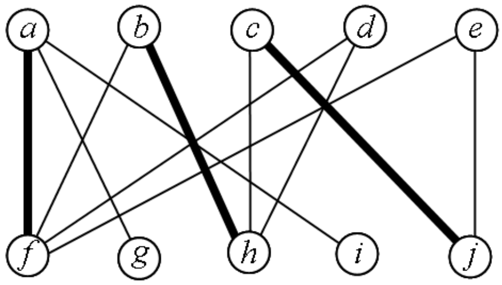
Step 2



Augmenting paths:



Augmented matching:

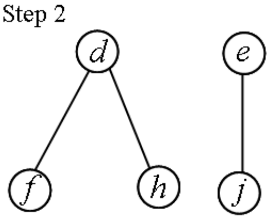


The choice of augmenting paths is not unique.

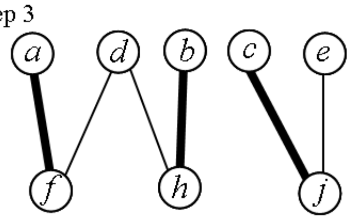
The **second** iteration is:

Step 1

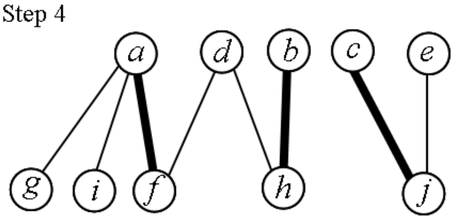
Step 2



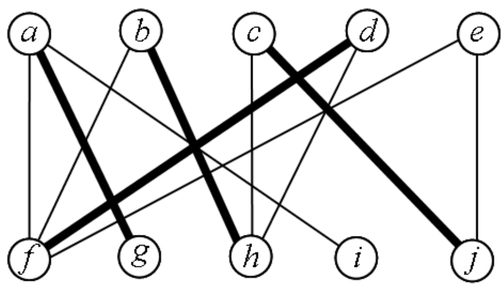
Step 3



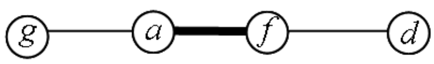
Step 4



Augmented matching:

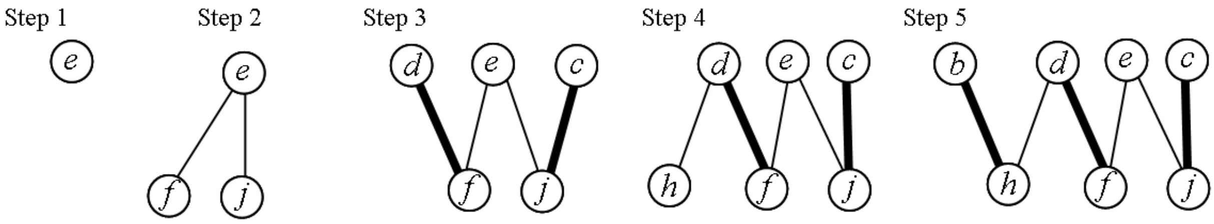


Augmenting path:



Again there is some choice in the way the alternating forest is constructed and augmenting path chosen.

The **third** iteration is:



Again there is some choice in the way the alternating forest is constructed. There is no augmenting path in the maximal alternating forest so the matching is maximum.

A minimum cover for the graph is given by $\{a\} \cup \{h, f, j\}$.