Solutions to question set for Chapter 1

1. A graph is called **3-regular** (or 'cubic') if every vertex has degree 3. Prove that a 3-regular graph has an even number of vertices.

   [Hint: use the Handshaking Lemma, given on p. 3 of the notes to Week 1, Lecture 3].

   SOLUTION: Let $G = (V(G), E(G))$ be a 3-regular graph. The Handshaking Lemma tells us that
   $$\sum_{v\in V(G)} d(v) = 2|E(G)|.$$
   Now since $d(v) = 3$ for every vertex v of G, this equality becomes $3|V(G)| = 2|E(G)|$ whence $|V(G)|$ must be divisible by 2. □

2. In Chapter 1, Section 3, the following pseudocode was given to calculate the value of $\mathrm{d}(v)$ for a vertex v:

   d(v) := 0
   for $e \in E$ **do**
       if $v \in e$ **then** d(v) := d(v) + 1

   Amend this pseudocode to make it correct when the input graph can include self-loops.

   SOLUTION: Self-loops at vertex v contribute 2 to $d(v)$. So we must test if e is a self-loop, i.e. if $|e| = 1$. The most obvious amendment is:

   d(v) := 0
   for $e \in E$ **do**
       if $v \in e$ **then**
           if $|e| = 1$ **then** d(v) := d(v) + 2 **else** d(v) := d(v) + 1

   A slightly more devious alternative you might have thought of is:

   d(v) := 0
   for $e \in E$ **do**
       if $v \in e$ **then** d(v) := d(v) + $\dfrac{2}{|e|}$

3. Prove the Handshaking Lemma by induction on the number of vertices.

   SOLUTION: We must show that the sum of the degrees is twice the number of edges for any graph $G = (V, E)$.

   Base case: we may start with $|V| = 0$ (this seems slightly sneaky but it is always a good idea to start with the lowest possible value — you just have to take care not to fall into a 'monochromatic cows' trap! [1]). Indeed, when $|V| = 0$ we have $2|E| = 2 \times 0 = 0 = \sum \mathrm{d}(v)$.

   Inductive Step: Assume the result is true for $|V| = K \geq 0$. Let G have $K+1$ vertices. We choose a vertex u and delete it, together with all incidence edges. This gives us a graph $G' = (V', E')$ on K vertices for which, by inductive hypothesis,
   $$\sum_{v\in V'} \mathrm{d}(v) = 2|E'|. \tag{1}$$

[1] E.g. see plus.maths.org/content/monochromatic-cows

Now replace the vertex u and replace one of the incident edges, say e. Clearly this increases the right-hand side of equation (**??**) by 2. If e is a self-loop at u then the left-hand side of equation (**??**) is increased by 2. If e is not a self-loop then the left-hand side of equation (**??**) is increased by one for each of its endpoints. So again the left-hand side is increased by 2. So equation (**??**) remains true when we replace a deleted edge. Replacing all the deleted edges gives us the required result for graph G.

The Handshaking Lemma now follows by induction. $\square$

4. The following pseudocode specifies an algorithm which checks if a graph G has an isolated vertex (that is, having zero degree).

```
Algorithm SeekIsolated
Input:    graph G = (V, E)
Output:   True if there is a vertex v ∈ V for which d(v) = 0; False otherwise
      foundzero:=False    # initialise to a False answer, then look for evidence to the contrary
1.    for v ∈ V do
          foundincidence:=False  # if this stays False in the following loop then
                                 # we've found a zero degree
2.        for e ∈ E do
3.            if v ∈ e then foundincidence:=True
          if not foundincidence then foundzero:=True
      return(foundzero)
end
```

Given an analysis of the complexity of this algorithm.

SOLUTION: The **for** loop at (1) takes $|V|$ steps to examine each vertex.

For each vertex the **for** loop at (2) takes $|E|$ steps to examine each vertex. After this loop finishes we take 1 step to test **if not** foundincidence.

So the total time is $|V| \times (|E| + 1)$ steps. Since $|V| \times (|E| + 1) = |V||E| + |V|$ and the 2nd term is less than the first we can write this as $O(|V||E|)$ steps.

5. (a) In Chapter 1, Section 3 the following Lemma was proved:

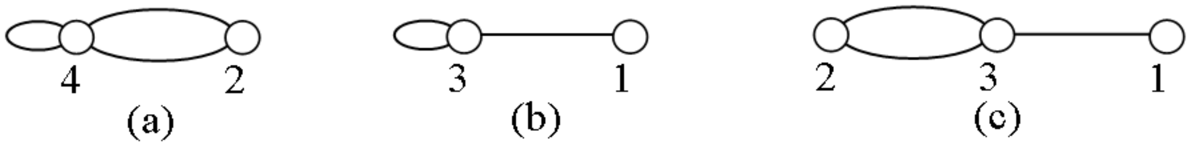
   Lemma If G is a simple graph on two or more vertices then G has two vertices with equal degree.

   Is this Lemma true for non-simple graphs? What about if self-loops are allowed but not multiple edges? What if multiple edges are allowed but not self-loops? Give counter examples as appropriate.

   (b) We could speed up the algorithm Stupid, given in Chapter 1, Section 3, by only checking all $\binom{|V|}{2}$ pairs of vertices. The **for** loop at 1. in the algorithm specification would still check every vertex but the **for** loop at 2. would only check vertices *further down the list*. What is the complexity of this new version of Stupid? Is it different from the original version when we use the big-Oh notation?

SOLUTION:

(b) Counterexamples for (a) non-simple, (b) loops only, and (c) multiple edges only are shown below:



(b) The question does not require pseudocode for the new version of Stupid, but here it is for the sake of completeness:

Algorithm **Stupid**
Input: simple graph $G = (V, E)$ with $V = \{v_1, v_2, \ldots, v_n\}$
Output: True if there are vertices $u, v \in V$ for which $\mathrm{d}(u) = \mathrm{d}(v)$; False otherwise
    Found:=False   # initialise the result
1.   **for** $u \in V$ **do**
        Let $u = v_i$, say.
2.   **for** $v \in \{v_{i+1}, \ldots, v_n\}$ **do**
3.   **if** $\mathrm{d}(u) = \mathrm{d}(v)$ **then** Found:=True
    return(Found)
end

Complexity: the **for** loop at (1) examines each vertex but each iteration of the loop will require fewer steps:

for the first vertex the **for** loop at (2) takes $|V| - 1$ steps to examine $v_2, \ldots v_n$;
for the second vertex the **for** loop at (2) takes $|V| - 2$ steps to examine $v_3, \ldots v_n$;
. . .
for the penultimate vertex the **for** loop at (2) takes 1 steps to examine v_n;
for the last vertex the **for** loop at (2) will not execute (no vertices remain to check).

So this is a total of $|V| - 1 + |V| - 2 + \ldots + 2 + 1 = \frac{1}{2}|V|(|V| - 1)$ iterations of the **for** loop at (2).

OR: just observe that $\binom{|V|}{2}$ pairs of vertices to check is $\frac{1}{2}|V|(|V|-1)$ checks. (also earns **8 marks**).

Now each iteration of the **for** loop at (2) uses $2|E|$ steps to compare $\mathrm{d}(u)$ and $\mathrm{d}(v)$.

So the total time complexity is $\frac{1}{2}|V|(|V| - 1) \times 2|E| = |V|^2|E| - |V||E|$ steps.

The complexity of the original version of Stupid was $|V|^2 \times 2|E|$ steps. In big-Oh notation we ignore multiples so this is $O(|V|^2|E|)$ (as explained in the lecture notes).

For the new version we ignore the subtracted 2nd term since it is smaller than the first term. This gives $O(|V|^2|E|)$.

So the new version has the same complexity, using big-Oh.