Chapter 9. Section 3

Kőnig's Theorem

The following lemma is given without proof:

Lemma Let G be a bipartite graph with parts X and Y. Let M be a matching in G and let F be a maximal M-alternating forest in G. Then

1. $U(F) = (X \backslash V(F)) \cup (Y \cap V(F))$ is a cover for G;
2. If F contains no M-augmenting path then $|U| = |M|$.

Recall that $\nu(G)$ denotes the size of a maximum matching in G and $\tau(G)$ denotes the size of a minimum cover of G.

Theorem (Kőnig, 1931) For a bipartite graph G $\nu(G) = \tau(G)$.

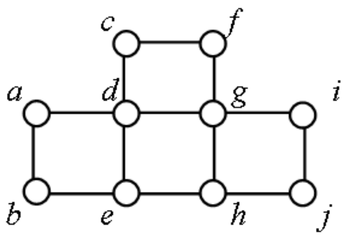
Proof: Execute Kőnig's algorithm on G. On termination, a matching M and a maximal M-alternating forest F have been found such that F contains no M-augmenting path. So by the Lemma, $U(F)$ is a cover for G and $|U(F)| = |M|$. Then by the Theorem of Chapter 8, Section 2, M is a maximum matching and $U(F)$ is a minimum cover. □

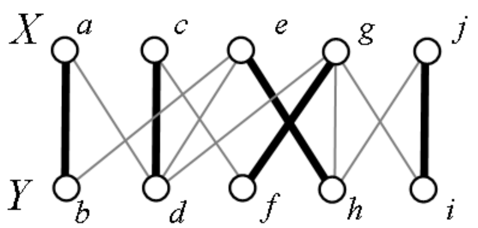
Corollary If G is a bipartite graph with parts X and Y and G has a perfect matching then X is a minimum cover for G.

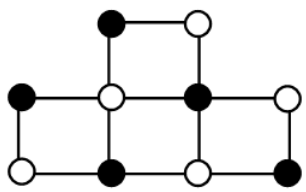
Proof: Let M be the perfect matching. By definition M is spanning, so X contains no M-unsaturated vertices. Then Kőnig's Algorithm will terminate immediately on G since the initial forest F contains no vertices. The cover $U(F)$ defined in the Lemma is: $U(F) = (X \backslash V(F)) \cup (Y \cap V(F)) = (X \backslash \emptyset) \cup (Y \cap \emptyset) = X$.

□

Examples: (1) The graph below left was given a minimum cover in the Chapter 8 Section 2. Now we can see how it was done: we identify the two parts X and Y which confirm that the graph is bipartite and find a perfect matching (the centre image). Then the Corollary tells us that X must be a cover, as indeed it is (the right-hand image).







(2) In the bipartite graph below left, the matching shown is maximum. To see this, run Kőnig's Algorithm. The result is the maximal M-alternating forest (in fact tree) shown below right. It contains no M-augmenting path, so the matching is maximum. The set $U(F)$ is calculated as

$$U(F) = (\{a, b, c, d, e\} \backslash \{a, A, c, C, d, e, E\}) \cup (\{A, B, C, D, E\} \cap \{A, c, C, d, e, E\}) = \{b\} \cup \{A, C, E\} = \{b, A, C, E\}.$$

You can check these four vertices are a cover: every edge of the graph on the left contains one of these vertices.