MTH6105 Algorithmic Graph Theory

Week 8. Lecture 2

Covers A **cover** in a graph $G = (V, E)$ is a subset U of the vertices of G satisfying

> *every edge contains at least one vertex of U.*

This definition has a certain dual symmetry with the definition of matching at the beginning of the Week 8 Lecture 1 notes. This turns out, in the case of bipartite graphs, to be genuine and gives us another minimax theorem.

An example of a cover is shown below left, with the convention that cover vertices are 'filled in'. We can think of a cover as a security problem: how many CCTV cameras do you need in order that every street is under surveillance? The cover U in the example is seen to be **minimal**: it cannot be made smaller (by removing some of its vertices — if we remove any camera then some street is unmonitored).

Is the cover **minimum**: meaning **NO** cover can be made smaller? Well, the cover above right has only 5 cameras and still every street is monitored. So our first cover was minimal but not maximal. *Again we have a problem which cannot be solved greedily.*

How do we know the right-hand cover is a minimum cover, if it is? This is where the relationship with matchings comes in...

Some terminology: let $\nu(G)$ (Greek letter 'nu') denote the size of a maximum matching in G; let $\tau(G)$ (Greek letter 'tau') denote the size of a minimum cover in G. We have

Lemma For any graph G the following inequality is satisfied:

$$\nu(G) \leq \tau(G). \tag{1}$$

Proof: Let M be a matching in G and let U be a cover. Every edge in M has some vertex in U by definition of cover, so we can make an assignment of matching edges to cover vertices. Suppose $|M| > |U|$, then by the Pigeon Hole Principle (see Week 2, Lecture 1) some cover vertex is assigned more than one matching edge. But this is impossible since by definition matching edges do not share vertices. So, for any matching M and any cover U, we have $|M| \leq |U|$. In particular this is true when M is a maximum matching and U is a minimum cover. The inequality follows. □

Theorem If a graph G has a matching M and a cover U and $|M| = |U|$ then M is a maximum matching and U is a minimum cover.

Proof: By definition $\nu(G) \geq |M|$ and $|U| \geq \tau(G)$. So $|M| = |U|$ implies $\nu(G) \geq \tau(G)$. The Lemma gives the reverse inequality. So equality holds and $|M| = \nu(G)$ (i.e. M is maximum) and $|U| = \tau(G)$ (i.e. U is minimum). □

We can now see that the right-hand cover in our example above was indeed minimum because the graph has a perfect matching which has 5 edges.