MTH6105 **Algorithmic Graph Theory**

Week 8. Lecture 1

Matchings in Graphs

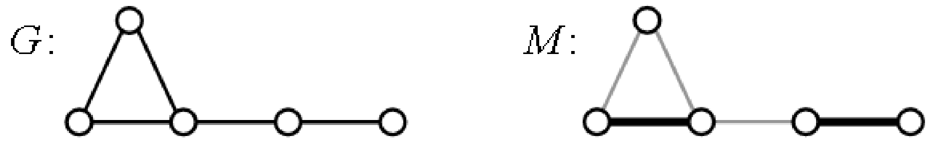
In a graph $G = (V, E)$, a **matching** is a subset of the edge set of G, say, $M \subseteq G$ satisfying

no vertex of G belongs to more than one edge of M.

In other words, we impose the degree condition:

$$d_M(v) \le 1 \text{ for all } v \text{ in } V. \tag{1}$$

An example is shown below, with the convention that edges of the matching M are shown in bold while non-matching edges are shown in grey.



Informally we see that a matching 'pairs up' vertices: each edge in the matching separates off its two endpoints.

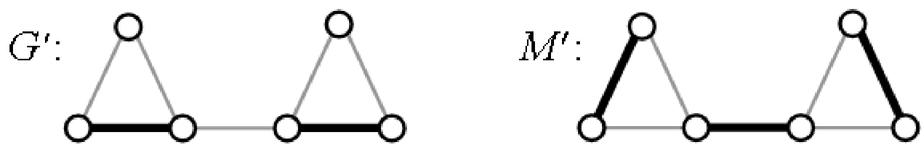
A matching M of G which is a spanning subgraph (i.e. contains every vertex of G) is called **perfect**.

The above matching is not perfect, and it cannot be because the graph G has 5 vertices and a perfect matching must include every vertex in a pair, which requires an **even** number of vertices. Indeed, any matching M must satisfy

$$|M| \le \frac{1}{2}|V|, \tag{2}$$

where $|M|$ denotes the number of edges in M. And a matching is perfect precisely when we have equality.

Not every matching will be perfect even when $|V|$ is even. To see this, add an extra vertex to our example, see below left:



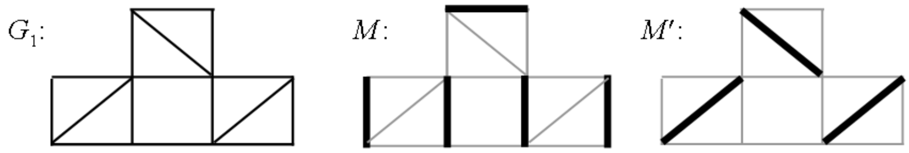
The graph G' is larger but M remains **maximal**: it cannot be extended without violating condition (1).

However, *maximal* does not imply *maximum* for matchings. The matching M' above right, has an extra edge and is perfect. Whenever maximal ≠ maximum we know at once that the greedy approach, which worked fine for minimum weight spanning trees and shortest paths, will not guarantee optimality.

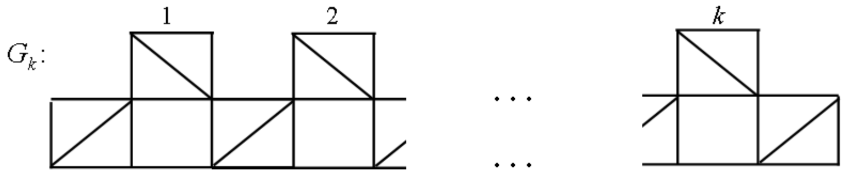
Remember: maxiMAL = THIS matching cannot be made bigger; maxiMUM = NO matching can be made bigger.

The matching M was maximal AND maximum in graph G; but in G' it is maximal but NOT maximum.

We will emphasise the potential difficulty of finding a maximum matching with another example:



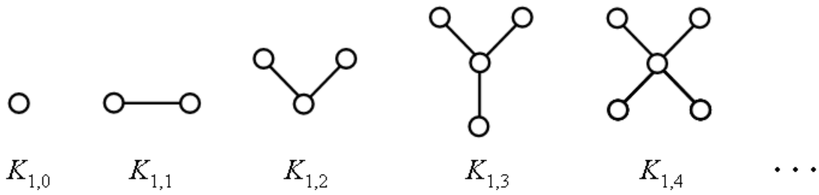
In this case M is a perfect matching for G_1 with $|V|/2 = 5$ edges, whereas M' is a maximal matching but has only 3 edges. We can extend this example to give an infinite family of graphs G_k which have a perfect matching but which contain maximal matchings with arbitrarily fewer edges than $|V|/2$:



So G_k is the graph with k 'upper squares'. **You do not have to remember these graphs for the MTH6105 exam!** There is nothing special about them; graph theorists invent infinite families of graphs whenever they need to show that a particular kind of behaviour is possible. In this case we want to show that G_k has a maximal matching with many fewer edges than the maximum possible.

It is easy to calculate that G_k has $16+11(k-1) = 11k+5$ edges and $10+6(k-1) = 6k+4$ vertices. It has a perfect matching, extending the one shown above for G_1 in the obvious way. This has $(6k+4)/2 = 3k+2$ vertices. And it has a maximal matching consisting of all the diagonal edges; and G_k has $2k+1$ diagonals. So the diagonal matching has $3k + 2 - (2k + 1) = k + 1$ fewer edges than the maximum possible.

One more example: we should establish that there are graphs on an even number of vertices which have no perfect matching. This is easy: the **star graphs** which are trees on $n + 1$ vertices, $n \geq 0$ with a central vertex of degree n can never have a matching with more than one edge:

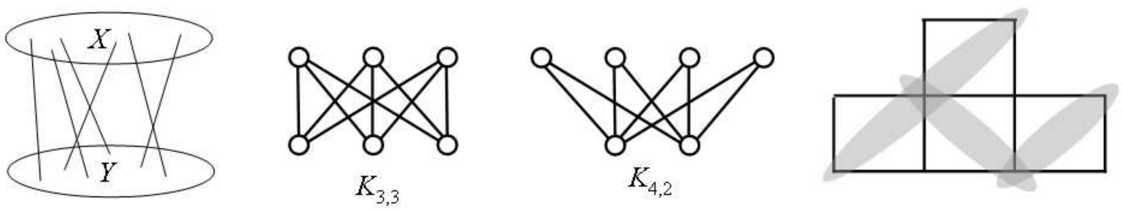


These graphs are denoted $K_{1,n}$; they are the so-called **complete bipartite graphs on 1 and *n* vertices**.

Bipartite graphs

A graph $G = (V, E)$ is called **bipartite** if there is a partition of V into sets X and Y (so $X \cup Y = V$ and $X \cap Y = \emptyset$) such that that every edge of G has an endpoint in each of X and Y; informally, all the edges 'go between' X and Y. The sets X and Y are called the *parts* of the graph G.

The image below, far left, illustrates this idea. Some actual examples are also given. $K_{3,3}$ is famous for being nonplanar (no drawing in the plane avoids edges which cross); however it does have a perfect matching (e.g. the vertical edges); $K_{2,4}$ does **not** have a perfect matching for the simple reason that its parts have different sizes. The graph on the right is also bipartite although it is not drawn in a way which exhibits its parts. One part is shown by shading.



Our interest in bipartite graphs lies in the fact that it is much easier to find maximum matchings in the case where a graph is bipartite.