Chapter 2. Section 1

Graph Theory: Yet More Definitions!

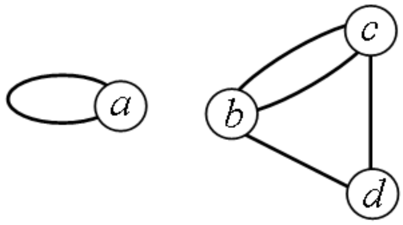
Connectivity

A **walk** is a consecutive sequence of vertices and incident edges. The walk is said to **join** the first and last vertex in the sequence. The **length** of the walk is the number of edges it contains.

E.g., in the following graph, a walk joining b and c might be

$$b,\ \{b,c\},\ c,\ \{c,d\},\ d,\ \{c,d\},\ c.$$

This walk has length 3. Notice that edges and vertices may be repeated. This walk is not quite well-defined as written since it is not specified which of the edges joining b and c is to be included. We can represent this walk much more conveniently, as bc, cd, cd.



More formally: we have a sequence of vertices $v_1, v_2, \ldots, v_k$, $k \geq 1$ and a sequence of edges $e_1, e_2, \ldots, e_{k-1}$, such that, for $1 \leq i \leq k-1$, $e_i = \{v_i, v_{i+1}$. If $k = 1$ then the walk contains zero edges (has length zero) and is said to be **trivial**.

E.g., in the above graph, we might have $k = 1$, $v_1 = a$. This is a trivial walk. Or we might have $k = 3$, $v_1 = v_2 = v_3 = a$, $e_1 = e_2 = \{a\}$. This would be the walk of length 2 which can be written as aa, aa.

A walk is called **closed** if it begins and ends with the same vertex; in our formal definition, $v_1 = v_k$.

E.g. Any trivial walk is closed. The walk aa, aa in the above graph is closed. The walk bc, cd, cd is not closed; however, it *contains* a closed walk: cd, cd.

A walk with no repeated vertices is called a **path**.

A closed walk with no repeated vertices except the first and last is called a **cycle**.

E.g. in the above graph aa is a closed walk of length 1 which is a cycle. The closed walk aa, aa is *not* a cycle since its sequence of vertices is a, a, a in which the second vertex is a repeat as well as the first and last. The closed walk bc, cd, bd is a cycle of length 3.

A graph is called **connected** if for every pair of vertices there is some walk joining these vertices.

E.g. the above graph is *not* connected since no walk joins a to b. If we delete vertex a then the remaining 3-vertex graph is connected.

A **tree** is a connected graph in which there are no cycles.

E.g. in the above graph, the edge subset $\{bd, cd\}$, together with vertex subset $\{b, c, d\}$, forms a tree. If instead our vertex subset was $\{a, b, c, d\}$ then we no longer have a tree: there are no cycles but this graph is not connected.

A **subgraph** of a graph $G = (V, E)$ is a graph (V', E') such that $V' \subseteq V$ and $E' \subseteq E$ and every edge in E' has both endpoints in V'.

E.g. $V' = \{a, b, c\}$, $E' = \{bc, bc\}$ is a subgraph of the above graph. $V' = \{a, b\}$, $E' = \{aa, bc\}$ is *not* a subgraph since $bc \in E'$ fails to have endpoint c in V'.