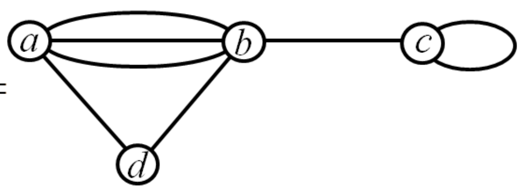
Chapter 7. Section 3

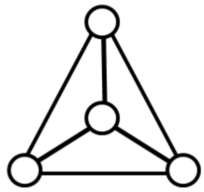
The Adjacency Matrix

We have seen graphs represented formally as a set and a collection of subsets; we have also seen them represented informally as drawings. A third representation uses a matrix.

If G is a graph with n vertices then the **adjacency matrix** $A(G)$ of G is an $n \times n$ matrix whose ij-th entry is equal to the number of edges joining vertex i to vertex j.

Two graphs are shown below together with their adjacency matrices.

$G =$ 



$$A(G) = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{pmatrix} 0 & 3 & 0 & 1 \\ 3 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix} \end{array} \qquad A(K_4) = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$$

The rows and columns of $A(G)$ are assumed to be indexed by the elements of the vertex set of G. It may help to label the rows and columns with the names of the corresponding vertices, as shown above left. In the matrix $A(K_4)$, no vertex names are given but the matrix still completely specifies a graph which is isomorphic to K_4.

Note:

1. $A(G)$ is a symmetric matrix: the i-th row is identical to the i-th column: this is because if vertex i has p edges to vertex j then vertex j must have p edges to vertex i.

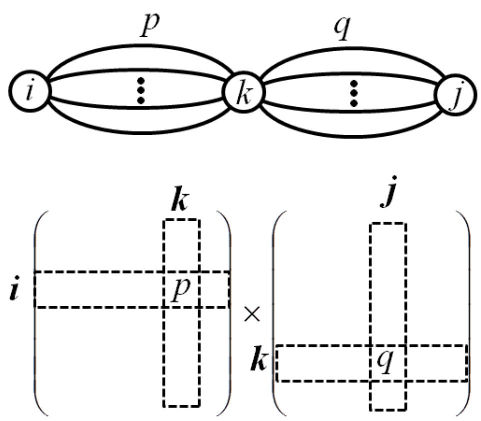
2. The degree $\mathrm{d}(v_i)$ of vertex v_i is given by the sum of the off-diagonal entries in row i + twice the diagonal entry.

We can discover much more about G from its adjacency matrix:

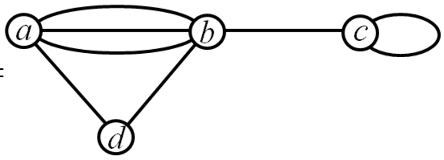
Theorem In a graph G the number of walks of length s from vertex i to vertex j is equal to the ij-th entry of $A(G)^s$.

Proof (outline) We see why the theorem is true for $s = 2$ and appeal to induction to give the general case.

Denote by a_{ij} the ij-th entry of $A(G)$. Then the ij-th entry in $A(G)^2$ is equal, by definition of matrix product, to $\sum_k a_{ik} \times a_{kj}$. But a_{ik} is the number of edges joining vertex i to vertex k; and a_{kj} is the number of edges joining vertex k to vertex j. So by the Product Principle, the summation is equal to the sum over all vertices k (possibly equal to i or j) of the number of walks of length 2 from i to j via k (see the diagram top right). And by the Summation Principle this is equal to the total number of walks of length 2 from i to j.



Example Below we show the graph from the previous page together with the square of its adjacency matrix:

$G =$ 

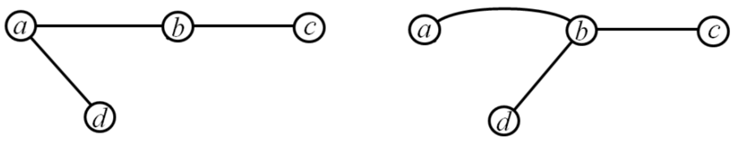
$$\begin{pmatrix} 0 & 3 & 0 & 1 \\ 3 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix} \times \begin{pmatrix} 0 & 3 & 0 & 1 \\ 3 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 10 & 1 & 3 & 3 \\ 1 & 11 & 1 & 3 \\ 3 & 1 & 2 & 1 \\ 3 & 3 & 1 & 2 \end{pmatrix}$$

The entries of the squared matrix can be checked against the graph: for instance, the 11 walks of length 2 from b to itself are: $9 = 3 \times 3$ walks via a, one walk bd, db and one walk bc, cb.

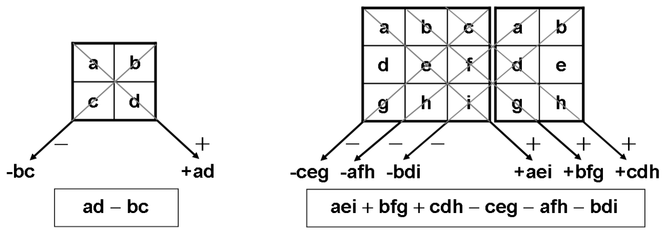
Counting Spanning Trees

In a graph G on n vertices there may be many subsets of $n-1$ edges which are trees on n vertices and are therefore spanning trees. We can use $A(G)$ to count exactly how many!

For the above graph, two spanning trees are shown below right; it is not hard to see that there are 7 in total: edge bc must be in every spanning tree while the self-loop at c cannot be in any and can be ignored. From the cycle abd each spanning tree must use 2 edges which contain all three vertices and this can happen in 7 ways.



Using $A(G)$ to derive this value of 7 involves the **determinant** function. Recall the simple rules for calculating determinants of 2×2 and 3×3 matrices as depicted on the right. At (a) we have $\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad-bc$; at (b) we have a similar summation over $3! = 6$ products, specified by repeating the first two columns of a given 3×3 matrix.[1]



(a)
(b)

Theorem (The Matrix Tree Theorem) Let G be a graph without self-loops. Let $A(G)$ be the adjacency matrix of G and let $D(G)$ be the matrix whose i-th diagonal element is the degree of vertex i and which is zero everywhere else. Let $L(G)$ be the matrix specified as $L(G) = D(G) - A(G)$ and denote by $L(G)(1\,|\,1)$ the matrix obtained from $L(G)$ by deleting the first row and column. Then

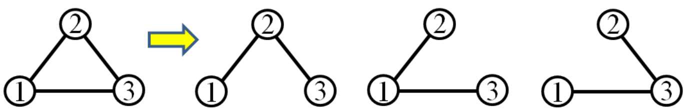
$$\#\text{spanning trees of } G = \det L(G)(1\,|\,1).$$

Example. For the graph at the top left of this page, ignoring the self-loop at vertex c, we find the degrees: $\mathrm{d}(a) = 4, \mathrm{d}(b) = 5, \mathrm{d}(c) = 1, \mathrm{d}(d) = 2$ and calculate:

$$A(G) = \begin{pmatrix} 0 & 3 & 0 & 1 \\ 3 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix},\ L(G) = \begin{pmatrix} 4 & -3 & 0 & -1 \\ -3 & 5 & -1 & -1 \\ 0 & -1 & 1 & 0 \\ -1 & -1 & 0 & 2 \end{pmatrix},\ L(G)(1\,|\,1) = \begin{pmatrix} 5 & -1 & -1 \\ -1 & 1 & 0 \\ -1 & 0 & 2 \end{pmatrix}.$$

and, using the Rule of Sarrus, #spanning trees$=5+0+0-1-0-2=7$.

Corollary (Cayley's Theorem) The number of labelled trees on n vertices is n^{n-2}.



The $3^{3-2} = 3$ labelled 3-vertex trees are shown above right as spanning trees of a labelled K_3.

[1]The illustration is reproduced from Whitty, R.W., *Operations Research and Combinatorial Optimisation*, University of London International Programmes, 2010. The 3×3 diagram is known as the Rule of Sarrus. A 4×4 version of the Rule of Sarrus, involving a summation over $4! = 24$ products can be found here: www.theoremoftheday.org/GeometryAndTrigonometry/Sarrus/Sarrus4x4.pdf.

Proof of Cayley's Theorem

The derivation of Cayley's Theorem as a corollary of the Matrix Tree Theorem involves a bit of work, but it is very neat and it produces another example of a non-linear recurrence relation. Here it is:

1. Denote by $t(n)$ the number of labelled trees on n vertices. Then $t(n)$ =#spanning trees in K_n (as illustrated for K_3 at the bottom of the previous page).
2. The matrix $L(G)$ for K_n is

$$L(K_n) = \begin{pmatrix} n-1 & -1 & \cdots & -1 \\ -1 & n-1 & & \vdots \\ \vdots & & \ddots & -1 \\ -1 & \cdots & -1 & n-1 \end{pmatrix} \quad \text{of size } n \times n.$$

3. Now by the Matrix Tree Theorem,

$$t(n) = \det L(G)(1\,|\,1) = \det \begin{pmatrix} n-1 & -1 & \cdots & -1 \\ -1 & n-1 & & \vdots \\ \vdots & & \ddots & -1 \\ -1 & \cdots & -1 & n-1 \end{pmatrix} \quad \text{(size now } (n-1)\times(n-1)\text{)}.$$

4. Now

$$\begin{aligned}
t(n) &= \det \begin{pmatrix} n-1 & -1 & \cdots & & -1 \\ -1 & n-1 & & & -1 \\ & & \ddots & & \vdots \\ \vdots & & & & -1 \\ -1 & -1 & \cdots & -1 & n-1 \end{pmatrix}, \quad \text{and apply Gaussian elimination:} \\
&= \frac{1}{(n-1)^{n-2}} \det \begin{pmatrix} n-1 & -1 & \cdots & & -1 \\ -(n-1) & (n-1)^2 & & & -(n-1) \\ & & \ddots & & \vdots \\ \vdots & & & & -(n-1) \\ -(n-1) & -(n-1) & \cdots & -(n-1) & (n-1)^2 \end{pmatrix} \\
&= \frac{1}{(n-1)^{n-2}} \det \begin{pmatrix} n-1 & -1 & \cdots & & -1 \\ 0 & (n-1)^2-1 & & & -(n-1)-1 \\ & & \ddots & & \vdots \\ \vdots & & & & -(n-1)-1 \\ 0 & -(n-1)-1 & \cdots & -(n-1)-1 & (n-1)^2-1 \end{pmatrix} \\
&= \frac{1}{(n-1)^{n-2}} \det \begin{pmatrix} n-1 & -1 & \cdots & & -1 \\ 0 & n(n-2) & & & -n \\ & & \ddots & & \vdots \\ \vdots & & & & -n \\ 0 & -n & \cdots & -n & n(n-2) \end{pmatrix} \\
&= (n-1)n^{n-2} \times \frac{1}{(n-1)^{n-2}} \det \begin{pmatrix} n-2 & -1 & \cdots & & -1 \\ -1 & n-2 & & & -1 \\ & & \ddots & & \vdots \\ \vdots & & & & -1 \\ -1 & -1 & \cdots & -1 & n-2 \end{pmatrix} = \frac{n^{n-2}}{(n-1)^{n-3}} t(n-1).
\end{aligned}$$

5. And finish with an inductive proof that $t(n) = \dfrac{n^{n-2}}{(n-1)^{n-3}} t(n-1)$ is solved by $t(n) = n^{n-2}$.