Chapter 7. Section 2

The following is easy to prove but has many useful consequences which are often not so obvious!

Lemma (The Handshaking Lemma)

For any graph $G = (V, E)$ we have

$$\sum_{v \in V} \mathrm{d}(v) = 2|E|.$$

The reason for the name is this: imagine a group of people. Some shake hands with each other, and this corresponds to an edge joining two vertices (people may shake hands with themselves, since we are not restricting our graph to be simple). Now the lemma says: count how many hands have been shaken in total, this must be equal to twice the number of handshakes.

Here is the same argument made formal:

Proof: Let us define a function f on pairs of vertices and edges:

$$f(v, e) = \begin{cases} 2 & \text{if } v \text{ and } e \text{ are incident and } e \text{ is a self-loop} \\ 1 & \text{if } v \text{ and } e \text{ are incident and } e \text{ is not a self-loop} \\ 0 & \text{if } v \text{ and } e \text{ are not incident} \end{cases}$$

Now for a given vertex v, $\sum_{e \in E} f(v, e) = \mathrm{d}(v)$. And for a given edge e, $\sum_{v \in V} f(v, e) = 2$. We count $f(v, e)$ over *all* vertices and edges. We can count by vertices and then edges or vice versa and get the same result:

$$\begin{aligned} \sum_{v \in V} \sum_{e \in E} f(v, e) &= \sum_{e \in E} \sum_{v \in V} f(v, e) \\ \text{so} \quad \sum_{v \in V} \mathrm{d}(v) &= \sum_{e \in E} 2 = 2|E|. \end{aligned}$$

$\square$

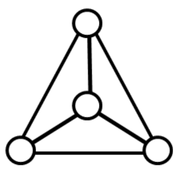
Corollary:

1. In a graph $G = (V, E)$ the number of vertices of odd degree is even.
2. Say that a graph G is k-**regular** if every vertex has degree equal to k. A graph of regular odd degree must have an even number of vertices.

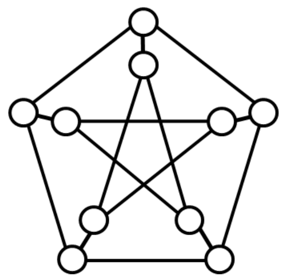
Proof.

1. The Handshaking Lemma says that the sum of the degree is an even number. So summing all odd degree must also give an even number, but the sum of an odd number of odd numbers is odd; so there must be an even number of odd degree vertices.
2. If G is regular of odd degree then all vertices have odd degree and therefore must be even in number by Part (1).

Graphs that are 3-regular (sometimes referred to as **cubic** graphs) are very well-studied. Here two famous examples:



K_4

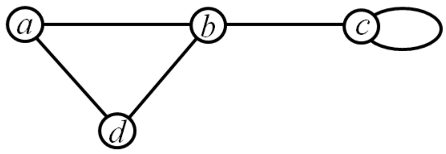


The Petersen Graph

Connectivity

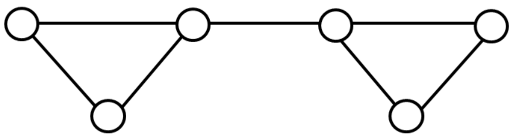
A **walk** in a graph G is a consecutive sequence of edges; this can be made formal by saying that two edges are **consecutive** if they contain at least one common vertex.

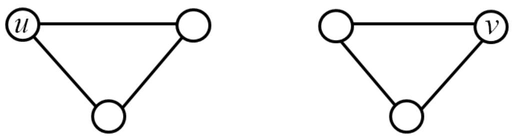
Example In the graph on the right *ab bc cc cb bd da* is a walk of length 6. Note that vertices or edges may be repeated. Note that *bc* and *cb* are the same edge: they are both equal to the subset $\{b, c\}$, being written in a different order merely to indicate the sequence of edges.



A graph is **connected** if, for any two vertices u and v of G, there is a walk containing both u and v.

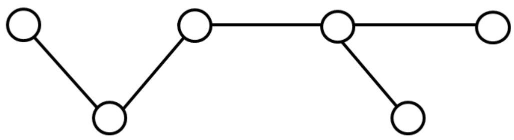
Of the two graphs below, the graph on the left is connected but the graph on the right is not, because no walk can contain both u and v.





A **cycle** is a walk which repeats precisely its first and last vertex. The graph at the top right contains a cycle of length 3, ab, bd, da, which is isomorphic to C_3 and a cycle of length 1: the self-loop at c.

A **tree** is a connected graph which contains no cycles. Two examples are shown below: the first is isomorphic to the path graph P_4, the second is isomorphic to a **subgraph** of the graph above left, i.e. a subset of its edges together with their incident vertices.



In a graph G, a **spanning tree** of G is a subgraph which (1) is a tree; and (2) contains all vertices of G. The tree above right is a spanning tree of the graph above left. Note that, since a tree must be connected, only a connected graph may possess a spanning tree.

Lemma A tree with n vertices has precisely $n - 1$ edges.

Proof By induction on n. The base case, $n = 1$, is true since K_1 is trivially connected and has no cycles, so it is a tree. And it has $0 = 1 - 1$ edges.

Assume the lemma is true for $n = K$ vertices. Let T be a tree with $K + 1$ vertices. Take two vertices, say u and v which are joined by a longest walk W having no repeated vertices. Then the degree of u satisfies $d(u) = 1$, since otherwise we can extend W from u to get a longer walk, or there must be and edge from u to another vertex of W forming a cycle. Now, if we delete u and its incident edge we cannot destroy connectivity, nor can we create a cycle, so we again have a tree. This has K vertices, so by inductive hypothesis it has $K - 1$ edges. We removed exactly one edge to derive this tree from T, so T has K edges, which is one fewer than the number of vertices. So the lemma is true for trees on K vertices, and the result follows by induction. □

The lemma tells us that a spanning tree is a subgraph which retains the least number of edges while preserving connectivity.