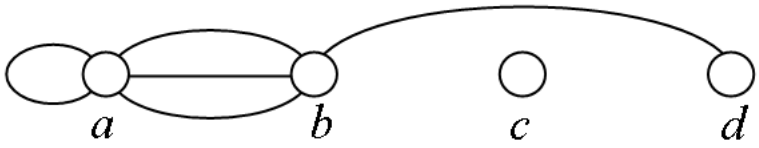
Chapter 7. Section 1

Graph Theory

Graph theory studies relationships between objects
using edges joining vertices.



A **graph** G is a pair (V, E) where V is a finite set and E is a collection of subsets of V of size at most 2.

Example: The above picture is a pictorial representation of the following graph:

$$\begin{aligned} V &= \{a, b, c, d\}, \\ E &= \big\{\{a\}, \{a, b\}, \{a, b\}, \{a, b\}, \{b, d\}\big\} \end{aligned} \tag{1}$$

Elements of V are called **vertices**.

Elements of E are called **edges**.

Elements of elements of E (the vertices of the sets of the collection) are called **endpoints**.

Example: In (1) the edge $\{a\}$ has one endpoint a; the edge $\{b, d\}$ has endpoints b and d.

Usually it is easier to write edges as pairs of endpoints; (1) becomes:

$$\begin{aligned} V &= \{a, b, c, d\}, \\ E &= \big\{aa,\ ab,\ ab,\ ab,\ bd\big\} \end{aligned}$$

Subsets of size 1 in E are called **self-loops**. In (1), $\{a\}$ is a self-loop.

Subsets in E which are repeated are called **multiple edges**. In (1), $\{a, b\}$ is an edge of multiplicity 3.

Graphs with neither self-loops nor multiple edges are called **simple**. In this case E is a *set* (rather than a collection) of sets of size exactly 2.

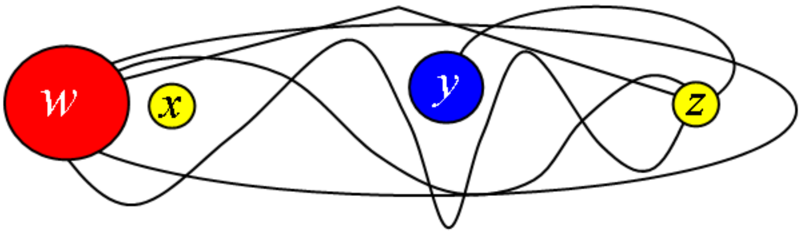
Two vertices are **adjacent** if they appear in the same subset of E. They are said to be **joined** by this edge.

An edge and its endpoints are said to be **incident** with each other. For example, in (1) a is incident with ab (three times); and bd is incident with b and with d.

The number of edges incident with a vertex v is the **degree** of v, denoted by $\mathrm{d}(v)$. Self-loops count twice. In (1) we have $\mathrm{d}(a) = 5$, $\mathrm{d}(b) = 4$, $\mathrm{d}(c) = 0$ and $\mathrm{d}(d) = 1$.

A vertex of degree zero is said to be **isolated**.

In many cases a pictorial representation of a graph (as used above) is just as good as the mathematical notation of sets and subsets. However, it is important to realise that this picture is not unique. For example, the following picture represents the same structure of vertex-edge incidences as the graph defined in (1):

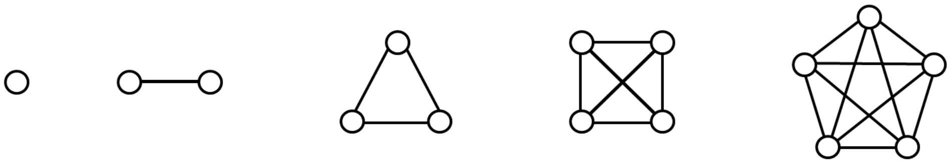


The graph represented here is not identical to (1) only in that the vertex names are different. The way in which the edges cross over each other may or may not be of interest (perhaps the graph is model of a road network); similarly the colours on the vertices may be important (perhaps they represent different countries). And the size of the vertices may mean something (populations?) However, these details are **not** part of the mathematical object which is the graph G.

Special Graphs

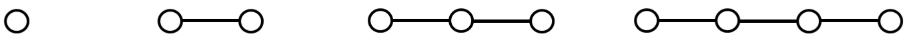
Some families of simple graphs occur very frequently in graph theory and deserve special names:

The complete graphs on n vertices, denoted by K_n, have all possible edges. Since K_n is simple this means the collection of edges is the set of all subsets of size 2, so K_n has $\binom{n}{2}$ edges. Graphically:



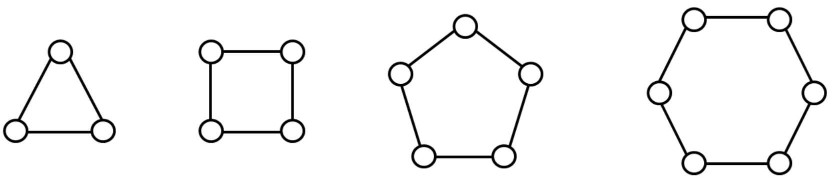
K_1 K_2 K_3 K_4 K_5

The path graphs on n vertices, denoted by P_n, consisting of a sequence of $n - 1$ edges. Graphically:



$P_1 \cong K_1$ $P_2 \cong K_2$ P_3 P_4

The cycle graphs on n vertices, denoted by C_n, consisting of a sequence of n edges, the first endpoint of the first edge being the same as the second endpoint of the last edge. Graphically:



$C_3 \cong K_3$ C_4 C_5 C_6

We have used the symbol $\cong$ to denote **isomorphism** (literally 'same structure') of graphs. As a further example we observe that the graphs on the right are isomorphic copies of K_4. One is a **plane** drawing of K_4, meaning that it has been drawn without edges intersecting other than at vertices.