# Chapter 5. Section 3

## Closed Form Definitions and Solutions

A **closed form** definition in mathematics is one which does not involve any sequences or series. This excludes $\sum$ and $\prod$ signs (and $\int$ and $\frac{d}{dx}$, but mostly we don't come across these). It also excludes recursive definitions as found in recurrence relations.

**Examples:**

1. $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$ is not a closed form definition but $\binom{n}{k} = \dfrac{n!}{(n-k)!k!}$ is closed (provided you don't regard $n!$ as implying a $\prod$ sign).

2. $(1+x)^{-1}$ is a closed form for $\sum_{k=0}^{\infty}(-1)^k x^k = 1 - x + x^2 - x^3 + \ldots$.

Here is an example which uses the binomial theorem in a more sophisticated way:

**Example** Find a closed form for

$$\sum_{k=0}^{\infty}\binom{n}{k}(1+k)x^k = 1 + 2nx + 3\binom{n}{2}x^2 + 4\binom{n}{3}x^3 + \ldots,$$

where $n$ and $k$ are positive integers.

**Solution.** We will solve this in two ways, one using clever manipulation of binomial expansions, the other using (very basic) calculus.

1. Expanding the factor of $(1+k)$:

$$\begin{aligned}
\sum_{k=0}^{\infty}\binom{n}{k}(k+1)x^k &= \sum_{k=0}^{\infty}\binom{n}{k}x^k + \sum_{k=0}^{\infty}\binom{n}{k}kx^k \\
&= (1+x)^n + \sum_{k=0}^{\infty}\binom{n}{k}kx^k \quad \text{(classic binomial expansion)} \\
&= (1+x)^n + \sum_{k=0}^{\infty}\frac{n!}{(n-k)!k!}kx^k \\
&= (1+x)^n + \sum_{k=0}^{\infty}\frac{n!}{(n-k)!(k-1)!}x^k \\
&= (1+x)^n + nx\sum_{k=1}^{\infty}\frac{(n-1)!}{((n-1)-(k-1))!(k-1)!}x^{k-1} \quad \text{(note change of lower limit!)} \\
&= (1+x)^n + nx\sum_{k=1}^{\infty}\binom{n-1}{k-1}x^{k-1}
\end{aligned}$$

which, if $K$ is substituted for $k-1$ (so $k=1$ becomes $K=0$) gives:

$$\begin{aligned}
&= (1+x)^n + nx\sum_{K=0}^{\infty}\binom{n-1}{K}x^K \\
&= (1+x)^n + nx(1+x)^{n-1} \quad \text{which is the required closed form.}
\end{aligned}$$

2. We start with $(1+x)^n$ and work forwards:

$$
\begin{aligned}
(1+x)^n = \sum_{k=0}^{\infty}\binom{n}{k}x^k &= \binom{n}{0}+\binom{n}{1}x+\binom{n}{2}x^2+\binom{n}{3}x^3+\ldots+\binom{n}{n}x^n \qquad (1)\\
\text{Differentiate both sides} \quad : & \\
n(1+x)^{n-1} &= 0+\binom{n}{1}+2\binom{n}{2}x+3\binom{n}{3}x^2+\ldots+n\binom{n}{n}x^{n-1}\\
xn(1+x)^{n-1} &= \binom{n}{1}x+2\binom{n}{2}x^2+3\binom{n}{3}x^3+\ldots+n\binom{n}{n}x^n \qquad (2)\\
(1)+(2):\ (1+x)^n+xn(1+x)^{n-1} &= \binom{n}{0}+2\binom{n}{1}x+3\binom{n}{3}x^2+\ldots+(n+1)\binom{n}{n}x^n\\
&= \sum_{k=0}^{\infty}\binom{n}{k}(1+k)x^k\\
\text{So:}\quad \sum_{k=0}^{\infty}\binom{n}{k}(1+k)x^k &= (1+x)^n+nx(1+x)^{n-1} \quad \text{is our closed form.}
\end{aligned}
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