Chapter 5. Section 2

The Binomial Theorem

The classic Binomial Theorem is the generating function for #subsets of size k in a set of size $n \geq k$:

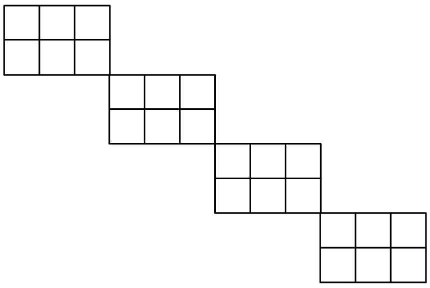
$$(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \ldots + \binom{n}{n-1}x^{n-1} + \binom{n}{n}x^n. \tag{1}$$

There is a 2-variable version which records both the subset chosen and the remaining non-chosen elements:

$$(X+Y)^n = \binom{n}{0}X^n + \binom{n}{1}X^{n-1}Y + \binom{n}{2}X^{n-2}Y^2 + \ldots + \binom{n}{n-1}XY^{n-1} + \binom{n}{n}Y^n,$$

and this allows coefficients to be extracted from powers of more complicated polynomials.

Example In how many ways may three non-attacking rooks be placed on the chessboard shown on the right?



Solution We know from Chapter 5 Section 1 that the rook polynomial for each of the four disjoint boards is $1+6x+6x^2$. Then the Product Principle tells us that the rook polynomial for rook placements on the union of these boards is $(1+6x+6x^2)^4$. Then the number of 3-rook placements will be the coefficient of x^3 in this product.

Put $X = 1+6x$ and $Y = 6x^2$. Then we want to know the coefficient of x^3 in

$$\begin{aligned}(X+Y)^4 &= \binom{4}{0}X^4 + \binom{4}{1}X^3Y + \binom{4}{2}X^2Y^2 + \ldots \\ &= (1+6x)^4 + 4(1+6x)^3\left(6x^2\right) + 6(1+6x)^2\left(6x^2\right)^2 + \ldots\end{aligned}$$

... but we may ignore the last term above and all following terms since they will involve only powers of x higher than 3. Our problem reduces to asking for the coefficient of x^3 in $(1+6x)^4 + 24x^2(1+6x)^3$. This returns us to the classic binomial theorem (1): in $(1+6x)^4$ the coefficient of x^3 is $\binom{4}{3}6^3 = 864$; in $(1+6x)^3$ the coefficient of x^1 (to combine with the $24x^2$) is $\binom{3}{1}6^1 = 18$. So the total coefficient for x^3 is $864 + 24 \times 18 = 1296$. So there are 1296 ways to place three non-attacking rooks on our chessboard.

There is another way in which we may generalise equation (1): replace the power n with any real (or even complex) number! This works by being specific about how we define binomial coefficients:

$$\binom{n}{k} = \frac{n \times (n-1) \times \cdots \times (n-k+1)}{k!}, \binom{n}{0} = 1, \binom{n}{k} = 0 \text{ if } k > n.$$

This is equivalent to our original definition (Week 1, Lecture 3) but avoids factorials in the numerator, which is therefore defined for any $n \in \mathbb{C}$.

Now we have an infinite generating function version of the binomial theorem:

$$(1+x)^n = \sum_{k=0}^{\infty}\binom{n}{k}x^k. \tag{2}$$

For example, setting $n = -1$ we can calculate: $\binom{-1}{0} = 1$, $\binom{-1}{1} = -1$, $\binom{-1}{2} = \frac{-1 \times -2}{2!} = 1$, etc. And

$$(1+x)^{-1} = \frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 - \ldots = \sum_{k=0}^{\infty}(-1)^k x^k.$$

We should note at once that writing a power series expansion for a function like $(1 + x)^{-1}$ does not in general provide a method for approximating the function. The infinite sum will converge for some values of x and not others. The binomial series, in its infinite form, is guaranteed to converge for $|x| < 1$. But putting $x = 10$, say, into the expansion of $(1 + x)^{-1}$ to get

$$(1 + 10)^{-1} = \frac{1}{11} = 1 - 10 + 10^2 - 10^3 + 10^4 - \ldots$$

is clearly a very dodgy thing to do![1]

Example: integer partitions If n is a positive integer then a **partition** of n is an expression of n as a sum of positive integers.

Thus for $n = 5$ we have

$$\begin{array}{cccc} 1+1+1+1+1 & 1+1+1+2 & 1+1+3 & 1+4 \\ & 2+2+1 & 2+3 & 5 \end{array}$$

Note: the order of summation does not matter: $2 + 3$ is the same partition as $3 + 2$. Counting **ordered** partitions, where these count as different, is equivalent to counting distributions of identical objects: see Week 2, Lecture 3.

Let $p(n)$ denote the number of (unordered) partitions of n. We just saw that $p(5) = 7$. We write down a generating function for partitions:

$$\begin{aligned} P(x) = \sum_{k=0}^{\infty} p(k)x^k &= p(0) + p(1)x + p(2)x^2 + p(3)x^3 + p(4)x^4 + p(5)x^5 + \ldots \\ &= 1 + x + 2x^2 + 3x^3 + 5x^4 + 7x^5 + 11x^6 + 15x^7 + 22x^8 + \ldots \end{aligned}$$

We have just seen that $(1 + x)^{-1} = 1 - x + x^2 - x^3 + \ldots$. by substituting $(-x)$ for x, we get

$$(1 - x)^{-1} = \frac{1}{1 - x} = 1 - (-x) + (-x)^2 - (-x)^3 + (-x)^4 - \ldots = 1 + x + x^2 + x^3 + x^4 + \ldots.$$

In the 1740s Leonhard Euler (pronounced ‘**Oil**-er’) invented generating functions to tackle the problem of evaluating $p(n)$ and discovered the following:

Theorem (Euler) $P(x) = \prod_{k=1}^{\infty} \frac{1}{1 - x^k} = \left(\frac{1}{1 - x}\right)\left(\frac{1}{1 - x^2}\right)\left(\frac{1}{1 - x^3}\right)\cdots.$

Proof. By the binomial theorem,

$$\left(\frac{1}{1 - x}\right)\left(\frac{1}{1 - x^2}\right)\left(\frac{1}{1 - x^3}\right)\cdots = \left(1 + x^1 + x^2 + \ldots\right)\left(1 + x^2 + x^4 + \ldots\right)\left(1 + x^3 + x^6 + \ldots\right)\cdots.$$

The right-hand side can be expressed as

$$\left(1 + x^1 + x^{1+1} + \ldots\right)\left(1 + x^2 + x^{2+2} + \ldots\right)\left(1 + x^3 + x^{3+3} + \ldots\right)\cdots. \tag{3}$$

Now expand the brackets: every partition of n will consist of a certain number (possibly zero) of 1s, a certain number of 2s etc. This will appear as $x^{1+1+\cdots} \times x^{2+2+\cdots} \times \cdots$, where the powers sum to n. So each of the $p(n)$ partitions of n is counted in our expansion. And no partition is counted more than once because, as expressed in (3), only the first bracket contributes 1s, only the second contributes 2s etc.

[1] Dodgy, but treated with mathematical care such ‘divergent series’ can be a powerful tool. There is a classic book about them: *Divergent Series* by G.H. Hardy, published in 1949, shortly after his death, and still in print.