Chapter 4. Section 3

More Recurrences and Bijections

Pascal's Rule again

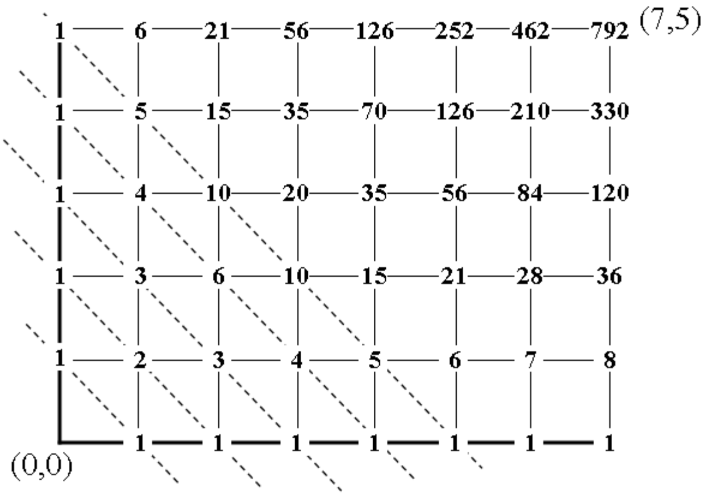
In Chapter 1, Section 3 we proved Pascal's Rule for binomial coefficients using the Summation Principle:

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}, n, k \geq 0, \binom{n}{0} = 1, \binom{n}{k} = 0 \text{ if } k > n. \tag{1}$$

We now recognise this as a recurrence relation. It is linear and homogenous; however, it is defined in terms of two variables, n and k, so we cannot apply our solving technique to it. Luckily we could prove a formula in terms of n and k: $\binom{n}{k} = \dfrac{n!}{(n-k)!k!}$.

In any case our interest is now to see the recurrence in action in combinatorial modelling, and to find **bijections** when it appears in models of **different structures**. One model was of subset selection: in fact, our method of proof of Pascal's Rule was precisely to derive it as a recurrence by constructing a choice from n out of two different choices from $n-1$.

So we know binomial coefficients count subsets of size k. Now here is something else they count, which we also met in Chapter 1: walks in the $n \times n$ grid. Our walks start at $(0,0)$ and proceed right **R** and upwards **U** in integer steps. We ask: how many ways can we arrive at point (n,k)? On the right, the first part of the integer lattice is shown with the answers written in. The diagonals, shown with dotted lines, are immediately recognisable as Pascal's Triangle. Indeed, we have:



Fact: the number of **RU** walks from $(0,0)$ to (n,k) is equal to $\binom{n+k}{n} = \binom{n+k}{k}$.

Proof: We argue inductively, the result clearly holding for small values of n and k. Since we must arrive at (n,k) with an **R** step or a **U** step, we have

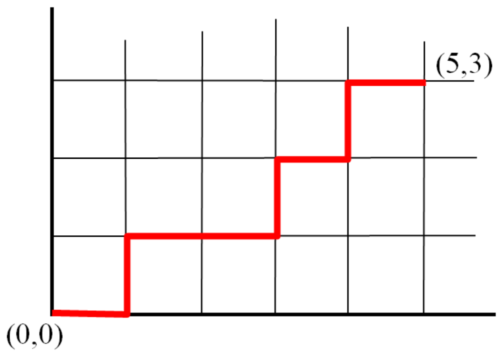
$$\begin{aligned}
\#\text{ways to reach } (n,k) &= \#\text{ways to reach } (n-1,k) + \#\text{ways to reach } (n,k-1) \\
&= \binom{(n-1)+k}{n-1} + \binom{n+(k-1)}{n} \quad \text{by inductive hypothesis} \\
&= \binom{n+k-1}{n+k-1-(n-1)} + \binom{n+k-1}{n+k-1-n} = \binom{n+k-1}{k} + \binom{n+k-1}{k-1} \\
&= \binom{n+k}{k} = \binom{n+k}{n} \quad \text{by Pascal's Rule.}
\end{aligned}$$

(This is an example of **double induction**: because our inductive hypothesis says the result is true for smaller values of both n an k. You are encouraged to spend a little time making the argument a bit more systematic and formal!)

Now we want a bijection because

$$\#\text{subsets of size } n \text{ chosen from a set of size } n+k = \#\mathbf{RU} \text{ walks to } (n,k).$$

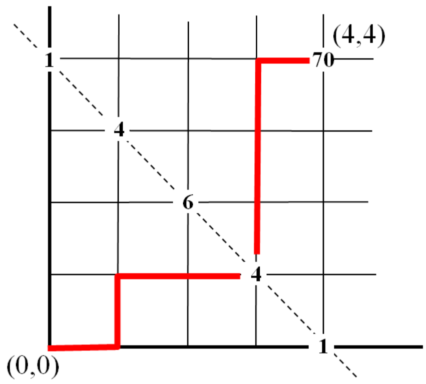
The bijection is easy to find: to get to (n,k) we must make $n+k$ steps, of which n must be **R** steps and k must be **U** steps. So we choose a subset of size n from $n+k$ for the **R** steps or, equivalently, a subset of size k from $n+k$ for the **U** steps.
E.g. suppose you give me the subset $\{1,3,4,6,8\}$ from $[8]$. I make eight steps of which the 1st, 3rd, 4th, 6th and 8th are **R** steps and the others are **U** steps: **RURRURUR**, and I arrive at $(5,3)$, as shown on the right.



We can use our bijection to prove a neat binomial coefficient identity:

$$\sum_{k=0}^{n}\binom{n}{k}^2 = \binom{2n}{n}.$$

Proof. We know that the number of **RU** walks from $(0,0)$ to (n,n) is $\binom{n+n}{n} = \binom{2n}{n}$. Each of these paths must pass the diagonal line passing through the points $(n-k,k), k = 0,\ldots,n$, leading from $(n,0)$ to $(0,n)$ and this is exactly the n row of Pascal's Triangle. The number of walks from $(0,0)$ to $(n-k,k)$ is $\binom{n-k+k}{k} = \binom{n}{k}$ and this is the same as the number of walks from $(n-k,k)$ to (n,n) by symmetry. So by the Product Principle the number of walks from $(0,0)$ to (n,n) passing through $(n-k,k)$ is $\binom{n}{k}^2$. Now the result follows by the Summation Principle.



An example pair of paths, for $n = 4, k = 1$, is shown above right.

The Catalan Numbers

Here is a sequence $(C_n)_{n\geq 0}$ which, it turns out, counts many different combinatorial structures:

$$1, 1, 2, 5, 14, 42, 132, 429, 1430, 4862, 16796, \ldots$$

The sequence, called the **Catalan numbers** after Euène Charles Catalan (1814–1894), can be defined in may different ways too. Here is a recurrence relation definition:

$$C_n = \sum_{k=0}^{n-1} C_k C_{n-k-1} = C_0C_{n-1} + C_1C_{n-2} + C_2C_{n-3} + \ldots + C_{n-1}C_0. \tag{2}$$

This is certainly not a linear recurrence: every term involves a product of two Catalan numbers. Nor does it have bounded degree, since it uses *all* previous terms of the sequence. However, we can guess a solution and then prove it:

$$C_n = \frac{1}{n+1}\binom{2n}{n}. \tag{3}$$

By the way, combining (3) with Pascal's Rule (equation (2)) gives another recurrence:

$$C_n = \frac{2(2n-1)}{n+1}C_{n-1},$$

which is still nonlinear (since the coefficient of C_{n-1} is not a constant) but *does* have bounded degree—so degree is not an invariant property of recurrence relations.