Chapter 4. Section 2

Combinatorial Modelling with Recurrences

The exam question in the previous section began by solving linear recurrences but ended by actually using one to model a 'real life' combinatorial structure (lines of square tiles). This is our real concern—no use solving equations if they don't do anything!

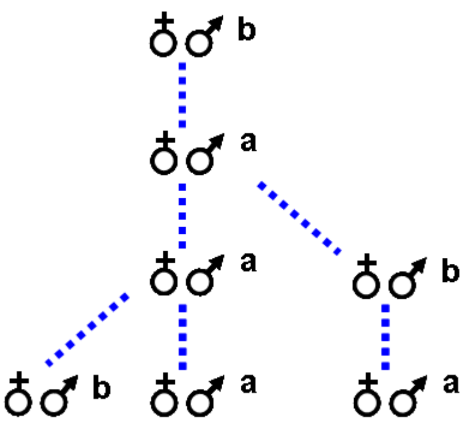
We shall count a number of different structures. We shall find that often structures of two different types lead to the same recurrence relation so that the counts are the same. Then we look for a way to match up the structures of one type with those of the other type: i.e. we look for a one-to-one correspondence or **bijection**. Finding bijections is at the heart of combinatorics.

Fibonacci's Rabbits

Leonardo de Pizza, better known as Fibonacci, asked the following question:

> I have a baby girl rabbit and a baby boy rabbit. After a month they are grown up and mate which leads, after one further month, to a new baby girl-boy pair. After another month they produce another pair; and by now their first offspring are grown up and mating. If all pairs of adult rabbits generate one new pair per month and no rabbits ever die, how many rabbits will there be in a year?

Fibonacci's question is illustrated on the right. Rabbit pairs are 'b' for 'baby' until they are 'a' for 'adult'. For each rabbit pair, from the month after it becomes adult, one new pair of baby rabbits appears.
We want to model the rabbit population growth combinatorially. This promises not only to answer Fibonacci's question but also to reveal connections to other structures.



Generation	Population (#pairs)
0	1
1	1
2	2
3	3

We know what the combinatorial model is, because it was the answer to part (e) of the exam question! Let F_n be the number of rabbit pairs at month n. So $F_0 = 1, F_1 = 1, F_2 = 2, F_3 = 3, \ldots$. What can we say about F_n in general? Well, in month n, $n \geq 2$, all the month $n-1$ rabbit pairs are still alive, so that is F_{n-1} pairs. And all the month $n-2$ pairs, whether baby or adult, will add exactly one new baby pair, so that is another F_{n-2} pairs. Total: $F_n = F_{n-1} + F_{n-2}$.

We can solve this to get a formula for F_n in terms of n, which will answer Fibonacci's question with a single calculation. Of course, we can determine F_n, for any n, by generating the complete sequence starting from F_0 and F_1:

0	1	2	3	4	5	6	7	8	9	10	11	12
1	1	2	3	5	8	13	21	34	55	89	144	233.

But a formula offers something extra: a 'long range' idea of how the population is growing.

$$
\begin{aligned}
\text{The homogenous equation:} \quad & F_n - F_{n-1} - F_{n-2} = 0 \\
\text{The characteristic equation:} \quad & r^2 - r - 1 = 0 \\
\text{Solutions (via quadratic formula):} \quad & r = \frac{1 \pm \sqrt{5}}{2} \\
\text{Homogeneous solution:} \quad & F_n = \alpha\left(\frac{1+\sqrt{5}}{2}\right)^n + \beta\left(\frac{1-\sqrt{5}}{2}\right)^n \\
\text{Solving for } \alpha \text{ and } \beta \text{ using } F_0 = F_1 = 1\text{:} \quad & \alpha = \frac{5+\sqrt{5}}{10},\ \beta = \frac{5-\sqrt{5}}{10} \\
\text{So:} \quad & {}_n = \left(\frac{5+\sqrt{5}}{10}\right)\left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{5-\sqrt{5}}{10}\right)\left(\frac{1-\sqrt{5}}{2}\right)^n \\
\text{which can be simplified to:} \quad & F_n = \frac{1}{\sqrt{5}}\left(\varphi^{n+1} - \frac{1}{(-\varphi)^{n+1}}\right), \qquad (1) \\
\text{where} \quad & \varphi = \frac{1+\sqrt{5}}{2}, \quad \text{which is the } \textbf{golden ratio} \approx 1.618.
\end{aligned}
$$

Since $\varphi > 1$, the $1/(-\varphi)^{n+1}$ term vanishes as $n \to \infty$. So now we know about long-term population growth: it is roughly $\left(\varphi/\sqrt{5}\right)\varphi^n \approx 0.72 \times 1.62^n$, as n grows large. Moreover, equation (1), gives a very simple ratio in the limit: $F_n/F_{n-1} \to \varphi$, as $n \to \infty$. So now we know about *relative* growth as well.

According to the exam question in Week 4, Lecture 1, F_n (aka G_n) also counts ways to make a line of tiles of length n from □ and ▭. Of course we ask WHY? The best possible answer is to invent a bijection from the F_n rabbit pairs to the F_n tile lines of length n. That is: give me a rabbit pair and I give you a corresponding line of tiles (an injection one way); and, I give you a line of tiles and you give me a corresponding rabbit pair (an injection the other way, making a bijection altogether).

We can almost get away with just drawing another picture! In the version of Fibonacci's rabbit tree below the pairs are represented by single vertices, labelled 'b' or 'a' as before. We map the month 1 adult pair to a square tile; in month 2 we add a square tile for the adult but replace with a rectangle for the baby. And we continue in the same way: down to an adult = add a square; down to a baby = replace right-most tile with a rectangle. This is not a very mathematical description of a bijection but it is enough to identify any line of tiles with a unique tree vertex.