Chapter 4. Section 1

Linear Recurrences: an Exam Question

These notes offer a complete solution to Question 2 from an old exam paper. Not exactly inspirational but useful for the reason that it's actually a nicely designed question which leads on to our next topic: using recurrences to answer combinatorial questions.

Question 2 Solve the following recurrence relations:

(a) $a_n = a_{n-1} + 6a_{n-2}, a_0 = 1,\ a_1 = 8.$

Solution: This is a homogeneous recurrence. So $a_n = h_n$ where h_n is derived from the roots of the characteristic equation.

The characteristic equation is $r^2 - r - 6 = 0$ which factorises: $(r-3)(r+2) = 0$, so $r_1 = 3$ and $r_2 = -2$.

Now $a_n = \alpha 3^n + \beta(-2)^n$ and we use the initial conditions to solve for α and β:

$$\begin{aligned} n = 0, a_0 = 1:\ 1 &= \alpha 3^0 + \beta(-2)^0 \quad \text{so } \alpha + \beta = 1 & (1)\\ n = 1, a_1 = 8:\ 8 &= \alpha 3^1 + \beta(-2)^1 \quad \text{so } 3\alpha - 2\alpha = 8 & (2)\\ 2\times(1) + (2):\ 5\alpha &= 10 \quad \text{so } \alpha = 2 \text{ and } \beta = -1. & (3) \end{aligned}$$

So $a_n = 2\,(3^n) - (-2)^n$.

(b) $a_n = 4a_{n-1} - 4a_{n-2}, a_0 = 3,\ a_a = 2.$

Solution: This is also a homogeneous recurrence.

The characteristic equation is $r^2 - 4r + 4 = 0$ which factorises: $(r-2)^2 = 0$, so $r_1 = 2$ and $r_2 = 2$.

Now $a_n = \alpha 2^n + \beta n(2)^n$ and we use the initial conditions to solve for α and β:

$$\begin{aligned} n = 0, a_0 = 3:\ 3 &= \alpha 2^0 + \beta \times 0 \times 2^0 \quad \text{so } \alpha = 3 & (4)\\ n = 1, a_1 = 2:\ 2 &= \alpha 2^1 + \beta \times 1 \times 2^1 \quad \text{so } \alpha + \beta = 1 \text{ and } \beta = 1 - 3 = -2. & (5) \end{aligned}$$

So $a_n = 3\,(2^n) - 2n2^n$ or $a_n = (3-2n)2^n$.

(c) $c_n = c_{n-1}^3,\ c_0 = 2.$

Solution: this is a nonlinear equation so we cannot derive the characteristic equation. Instead we test values until we guess a formula for c_n. Then we prove it is correct by induction on n.

Now $c_1 = c_0^3 = 2^3$, $c_2 = c_1^3 = \left(2^3\right)^3 = 2^{3\times 3} = 2^{3^2}$, $c_3 = c_2^3 = \left(2^{3^2}\right)^3 = 2^{3\times 3^2} = 2^{3^3}$.

We can spot the pattern and guess $c_n = 2^{3^n}$.

We prove this by induction on n:

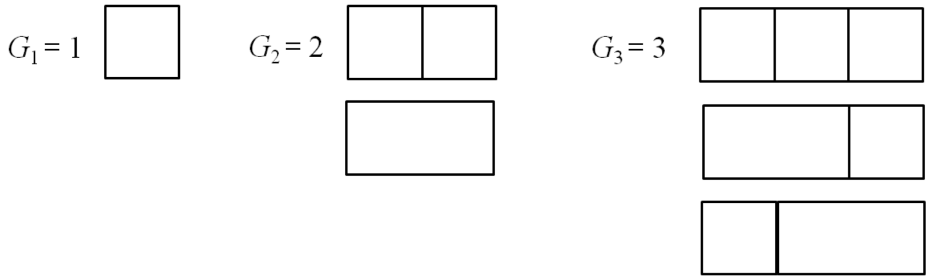
For $n = 0$, $2^{3^0} = 2^1 = 2 = c_0$ so the result is true for $n = 0$.

Suppose the result is true for $n = K$. Then

$$c_K^3 = \left(2^{3^K}\right)^3 = 2^{3\times 3^K} = 2^{3^{K+1}} = c_{K+1},$$

so the result is true for $n = K + 1$ and therefore follows in general.

For the final part of the question you are given an unlimited supply of two tiles as shown below and asked how many ways, denoted G_n, a line of tiles of length n can be made. The values of $G_1 = 1$, $G_2 = 2$ and $G_3 = 3$ are given as shown:



(d) Prove that $G_n = G_{n-1} + G_{n-2}$ for $n \geq 3$.

Solution: Take a line of length n and look at the final tile: it must be either a 1-tile or a 2-tile.

If it is a 1-tile then what comes before is a line of length $n - 1$ and this can be made in G_{n-1} ways by definition.

If it is a 2-tiles then what comes before is a line of length $n - 2$ and this can be made in G_{n-2}.

Now by the Summation Principle $G_n = G_{n-1} + G_{n-2}$.

(e) If G_0 is defined to have value 1, then what is the usual name for the sequence $(G_0)_{n\geq 0}$?

Solution: The Fibonacci sequence.

And we shall look at this as an example of combinatorial modelling next.