Chapter 3. Section 3

Solving Linear Recurrences

For our purposes, a recurrence relation of degree m will consist of

1. A definition, $a_n = f(a_{n-1}, a_{n-1}, \ldots, a_{n-m}, n)$, of a_n in terms of $a_{n-k}, 1 \leq k \leq m$, and n itself, where f is some function of $m+1$ variables;

2. Initial conditions specifying the first m sequence values $a_0, a_1, \ldots, a_{m-1}$.

For example, for the factorial function $n!$ is defined as $f((n-1)!, n)$, with the function f just multiplying its two arguments together to give $n \times (n-1)!$.

Our aim is to find a different function which gives a_n just in terms of n. For example, $n! = \prod_{k=1}^{n} k$. This is called **solving** the recurrence relation. In general the solution will incorporate the initial conditions: for example $n! = \prod_{k=1}^{n} k$ automatically gives $0! = 1$ because, by convention, an empty product has value 1.

We cannot hope to solve an arbitrary recurrence relation. Here is an example to convince you:[1]

$$a_n = a_{n-1} + \gcd(a_{n-1}, n+1), n \geq 1, \text{ with initial condition } a_0 = 7,$$

where $\gcd(x, y)$ is the greatest common divisor function. This is a recurrence whose degree is just 1, but it appears hopeless to look for a formula for a_n in terms of n. The reason? Because the sequence of values of $\gcd(a_{n-1}, n+1)$ consists entirely of 1's and prime numbers:

$$1, 1, 1, 5, 3, 1, 1, 1, 1, 11, 3, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 23, 3, 1, 1, \ldots$$

and the construction of prime-producing formulae in a single variable n appears beyond the scope of current mathematics.[2]

What we *can* solve are **linear recurrences** in which the a_k are only involved in the form of a linear sum with **constant coefficients**:

$$a_n = f(n) + \sum_{k=n-m}^{n-1} c_k a_k,$$

where c_k are constants, f is a function of n, and initial conditions are supplied for $a_0, a_1, \ldots, a_{m-1}$.

Typical examples might be:

$$\begin{aligned} F_n &= F_{n-1} + F_{n-2}, n \geq 2, a_0 = a_1 = 1, \text{ which is the Fibonacci sequence, or} \\ a_n &= a_{n-1} + 2a_{n-2} - 5a_{n-3} + 3n - 7, n \geq 3, a_0 = 1, a_1 = 3, a_4 = 17 \text{ which is made up.} \end{aligned}$$

We will demonstrate how linear recurrences are solved by way of a complete worked example. This is obviously an approach which is lacking in theoretical background! However, (1) we will meet the problem again in a more theoretical light when we work with generating functions, and (2) it will allow us to recall a close link to solving linear differential equations, giving us a two-for-the-price-of-one technique!

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[1] This recurrence was discovered very recently at a summer school for university students. You can find out more at recursed.blogspot.co.uk/2008/07/rutgers-graduate-student-finds-new.html.

[2] Although there are hugely complicated (and unusable) polynomials in many variables which are known to do the job, e.g. see primes.utm.edu/glossary/xpage/MatijasevicPoly.html.

Example We will solve the recurrence relation

$$a_n = 3a_{n-1} - 4a_{n-3} + 6n - 9, n \geq 3 \quad (1)$$

$$\text{initial conditions: } a_0 = -1,\ a_1 = 5,\ a_2 = 18, \quad (2)$$

assuming, without justifying it, that any solution has the form $a_n = b_n + h_n$, where

b_n solves the recurrence (1) but not necessarily the initial conditions (2); and

h_n solves the **homogeneous equation** obtained by ignoring all terms of (1) except those involving the a_k.

1. Although it is not part of the solution it is a good idea to find a few more values of the sequence $(a_k)_{k\geq 0}$; this will allow us to check our solution at the end.

$$\begin{aligned} a_3 &= 3a_1 - 4a_0 + 6 \times 3 - 9 = 3 \times 18 - 4 \times -1 + 9 = 67; \\ a_4 &= 3a_3 - 4a_1 + 6 \times 4 - 9 = 3 \times 67 - 4 \times 5 + 15 = 196. \end{aligned}$$

2. Now we solve for b_n in (1) assuming, again without justifying it, that our solution will have the form $b_n = pn + q$.

$$\begin{aligned} pn + q &= 3(p(n-2) + q) - 4(p(n-3) + q) + 6n - 9 \\ \text{so } 0 &= -pn + 3pn - 4pn + 6n \\ &\quad -q - 3p + 3q + 12p - 4q - 9 \\ \text{equate coefficients: } 0 &= -29 + 6 \Rightarrow p = 3 \\ \text{and } 0 &= -2q + 9p - 9 = 2q + 9 \times 3 - 9 \Rightarrow 0 = -2q + 18 \Rightarrow q = 9. \end{aligned}$$

So $p = 3$, $q = 9$ and $b_n = 3n + 9$.

3. Now we solve for h_n in (2), where h_n is the **homogeneous equation** $a_n = 3a_{n-1} - 4a_{n-3}$, which we will rewrite as $a_n - 3a_{n-1} + 4a_{n-3} = 0$. We will assume, without justification, that $a_n = r^n$ solves this:

$$\begin{aligned} r^n - 3r^{n-1} + 4r^{n-3} &= 0. \quad \text{Divide through by lowest power of } r\text{, which is } r^{n-3}: \\ r^3 - 3r^2 + 4 &= 0. \quad \text{This is called the } \textbf{characteristic equation.} \end{aligned}$$

Solve for r by factorising:

$r^3 - 3r^2 + 4 = (r+1)(r-2)^2$ (Factor Theorem: $(-1)^3 - 3(-1) + 4 = 2^3 - 3(2^3) + 4 = 0$),

so we have a root $r = -1$ and a **repeated root** $r = 2$.

4. Now the solution method continues by asserting (again this needs justification) that

$$h_n = \alpha(-1)^n + (\beta + \gamma n)\, 2^n. \quad (3)$$

In general, a root r of the characteristic equation which appears with multiplicity s contributes to h_n a sum of terms $\left(\sum_{i=0}^{s-1} \alpha_i n^i\right) r^n$. In this case, the root $r = 2$ contributes $(\beta + \gamma n)\, 2^n$ since it has multiplicity $s = 2$.

5. We solve for α, β and γ by evaluating $h_0 + b_0, h_1 + b_1$ and $h_2 + b_2$ and equating to $a_0 = 1, a_1 = 6$ and $a_2 = 12$:

$$\begin{aligned}
n = 0,\ a_0 = -1: &\quad \alpha(-1)^0 + (\beta + \gamma \times 0)\,2^0 + 3 \times 0 + 9 = -1 \\
&\quad \alpha + \beta = -10 && (4) \\
n = 1,\ a_1 = 5: &\quad \alpha(-1)^1 + (\beta + \gamma \times 1)\,2^1 + 3 \times 1 + 9 = 5 \\
&\quad -\alpha + 2\beta + 2\gamma = -7 && (5) \\
n = 2,\ a_2 = 18: &\quad \alpha(-1)^2 + (\beta + \gamma \times 2)\,2^3 + 3 \times 2 + 9 = 18 \\
&\quad \alpha + 4\beta + 8\gamma = 3 && (6) \\
\text{Now} & \\
(4) + (5): &\quad 3\beta + 2\gamma = -17 && (7) \\
(5) + (6): &\quad 6\beta + 10\gamma = -4 && (8) \\
(8) - 2 \times (7): &\quad 6\gamma = 30. && (9)
\end{aligned}$$

So $\gamma = 5$, giving $\beta = -9$ and finally $\alpha = -1$.

So our solution to (1) is

$$\begin{aligned}
a_n &= -1 \times (-1)^n + (-9 + 5n)\,2^n + 3n + 9 \\
&= (-1)^{n+1} + (5n - 9)2^n + 3n + 9 && (10)
\end{aligned}$$

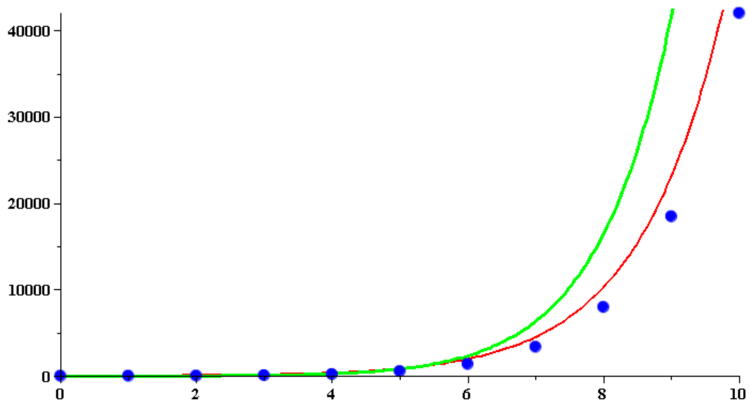
6. Checking back against our earlier calculation for $n = 3, 4$:

$$\begin{aligned}
a_3 &= (-1)^4 + (5 \times 3 - 9)2^3 + 3 \times 3 + 9 = 67; \\
a_4 &= (-1)^5 + (5 \times 4 - 9)2^4 + 3 \times 4 + 9 = 196.
\end{aligned}$$

Of course we have not gained anything in terms of calculating a_n for small values of n. Perhaps it is an advantage that we can now find the value of a_{1000}, say, without having to use the recurrence to work through a_0 up to a_{999} first.

But a much more real reward is that we now understand the long-term **growth** of a_n. As $n \to \infty$ all terms in equation (10) become insignificant except $5n2^n$. So we know that our recurrence is behaving like an $n2^n$ 'thing', as opposed to an $n3^n$ thing or an $n^2 2^n$ thing. This is impossible to detect directly from the original recurrence relation.

The plot below shows the exact values of a_n (the blue circles) against $y = 5x2^x$ (the red curve) and $x^2 2^x$ (the green curve). You can see that $5x2^x$ is a pretty good match for our actual sequence.



Plots of $y = 5x2^x$ (red, lower curve), $y = x^2 2^x$ (green, upper curve) and a_n (blue circles).

A Connection Linear ordinary differential equations may be solved following exactly the same procedure as we followed above. It seems worth following through the ODE version of the same example, just because it is so impressive that the same technique works across the apparently huge divide between continuous mathematics (analysis) and discrete mathematics (combinatorics).

Example Solve the 3rd-order ordinary linear differential equation

$$\frac{d^3y}{dx^3} - 3\frac{d^2y}{dx^2} + 4y - 6x + 9 = 0,$$

subject to initial conditions $y = -1$ at $x = 0$, $y = 5$ at $x = 1$, $y = 18$ at $x = 2$.

First we find a **particular solution** of the form $y_b = px + q$.

Since $\dfrac{d^3y_b}{dx^3}$ and $\dfrac{d^2y_b}{dx^2}$ are both zero, substituting $b(x)$ into our ODE gives $4(px+q)-6x+9 = 0$ and, equating coefficients, $4p - 6 = 0$ and $4q + 9 = 0$, giving $p = 3/2$ and $q = -9/4$. So $y_b = \dfrac{3}{2}x - \dfrac{9}{4}$.

Now write down the characteristic equation: $r^3 - 3r^2 + 4 = 0$. It has solution $r = -1$ and repeated solution $r = 2$. This means that the **general solution** has the form

$$y = y_h + y_b = \alpha e^{-x} + (\beta + \gamma x)e^{2x} + \frac{3}{2}x - \frac{9}{4}.$$

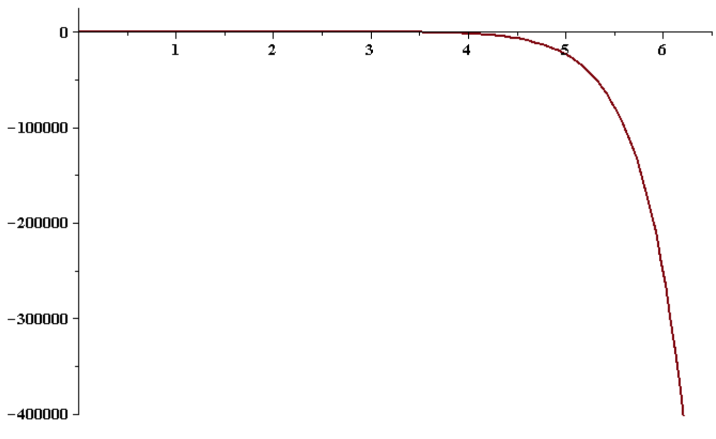
We can now use the initial conditions to solve for α, β and γ. We should not expect the equations to solve as easily as for the recurrence relation—they involve negative exponentials. Here is what Maple gives as an exact solution:

$$\begin{aligned} y \;=\; & \frac{-1}{4\,(e^8 - 2e^5 + e^2)}\Big(23(x-2)\,e^{2x+6} - 5(x-2x)\,e^{2x+5} + 5(x-1)\,e^{2x+2} - 69(x-1)\,e^{2x+4} \\ & -5\,e^{8-x} + 46\,e^{6-x} - 69\,e^{4-x} + 69x\,e^{2x+1} - 23x\,e^{2x} - 3(2x-3)\left(e^8 - 2\,e^5 + e^2\right)\Big). \end{aligned}$$

The approximate solution is a good deal easier to look at:

$$y\,(x) = \frac{3}{2}x - \frac{9}{4} + 0.01062018522\,e^{-x} + (1.239379815 - 0.4617306850\,x)\,e^{2x}.$$

Why not plot this too? I invite you to speculate about the relationship of this plot to the last!



Plot of $y\,(x) = \frac{3}{2}x - \frac{9}{4} + 0.01062018522\,e^{-x} + 1.239379815\,e^{2x} - 0.4617306850\,xe^{2x}$ for $0 \le x \le 10$.