Chapter 3. Section 2

Recurrence Relations

If, instead of having a function $f : [m] \longrightarrow [n]$ we have $f : \mathbb{Z}^{\geq 0} \longrightarrow \mathbb{Z}$ then we can think of the function values as forming an infinite sequence $f(0), f(1), f(2)\ldots$. More conveniently, we write $f_0, f_1, f_2, \ldots$. And more usually, perhaps to make clear the distinction from arbitrary functions we have

$$a_0, a_1, a_2, \ldots, \text{ or } (a_k)_{k\geq 0}\,.$$

The terms of the sequence may be defined by a rule. For example $\left(2^k\right)_{k\geq 0}$ is the sequence of powers of 2: $1, 2, 4, 8, 16, 32, \ldots$; and $\left((-1)^k(2k+1)\right)_{k\geq 0}$ is the sequence of odd numbers with alternating signs: $1, -3, 5, -7, \ldots$.

Two classic sequences are

Arithmetic progression (AP) having the form $a_k = a_0 + kd$, $k \geq 0$, where d is called the **common difference** and a_0 is called the **initial term**. It is easy to derive a formula for the **sum of the terms** of an AP up to $k = n$ as

$$\sum_{k=0}^{n}(a_0+kd) = \sum_{k=0}^{n} a_0+\sum_{k=1}^{n} kd = (n+1)a_0+d\sum_{k=1}^{n} d = (n+1)a_0+d\times\frac{1}{2}n(n+1) = \frac{1}{2}(n+1)(2a_0+nd).$$

Geometric progression (GP) having the form $a_k = a_0r^k$, $k \geq 0$, where r is called the **common ratio** and a_0 is called the **initial term**. It is easy to derive a formula for the sum of the terms of a GP up to $k = n$ by noticing that $(1 + r + \ldots + r^n)(1 - r) = 1 - r^{n+1}$, so that

$$\sum_{k=0}^{n} a_o r^k = a_0\,(1 + r + \ldots + r^n) = a_0\left(\frac{1 - r^{n+1}}{1 - r}\right),$$

with the limit, for $|r| < 1$, being $a_0/(1 - r)$.

Both APs and GPs define their general term a_k in terms of a_0. The value of a_0 has to be specified, of course, as an **initial condition**. We can extend this:

> A **recurrence relation** is an infinite sequence $(a_k)_{k\geq 0}$ whose n-th term a_n is defined in terms of previous terms a_k, $0 \leq k < n$. If the term of lowest index in the definition is a_{n-m} for some fixed finite value of m, then m is called the **degree** of the relation, and if **initial conditions** specify the values of $a_0, a_1, \ldots a_{m-1}$, then all values of the sequence are specified.

We can think of a recurrence relation of finite degree m as being a window of length m which slides along the sequence. At any point, all values in the window are known, and this allows the next value to be calculated. Then the window can move along one to the right and the calculation can be repeated.

$$\boxed{a_0, a_1, a_2,}\ a_3, a_4 \ldots, a_{n-5}, a_{n-4},\ \boxed{a_{n-3}, a_{n-2}, a_{n-1},}\ a_n, a_{n+1}, \ldots$$

Examples

1. The **factorial** function $n!$ is a sequence $1, 1, 2, 6, 24, 120, 720, 5040, \ldots = (a_k)_{k\geq 0}$, defined by $a_n = na_{n-1}$, with initial condition $a_0 = 0! = 1$. This has degree 1.

2. The **Fibonacci sequence**, $1, 1, 2, 3, 5, 8, 13, 21, \ldots$, in which each term is the sum of the two previous is defined by $a_n = a_{n-1} + a_{n-2}$, with initial conditions $a_0 = a_1 = 1$. This has degree 2.