Chapter 3. Section 1

Functions

We usually restrict attention to a **domain** $[m] = \{1, 2, \ldots, m\}$ and a **codomain** $[n] = \{1, 2, \ldots, n\}$. A function $f : [m] \longrightarrow [n]$ assigns to each element of $[m]$ a unique element of $[n]$.

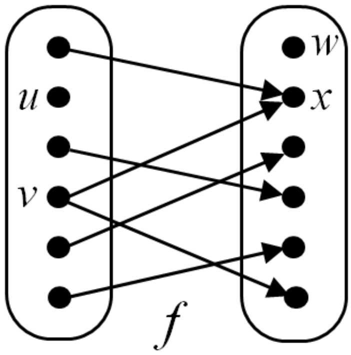
The function f may be

One-to-one (or **1–1** or **injective** or **an injection**) when, for all $x, y \in [m]$ $f(x) = f(y)$ if and only if $x = y$.

Onto (or **surjective** or **a surjection**) when, for all $y \in [n]$, $f(x) = y$ for some $x \in [m]$.

Bijective (or **a bijection** or **a one-to-one correspondence**) when f is both one-to-one and onto.

A **mapping** $f : [6] \longrightarrow [6]$ which fails to be a function because not every element is assigned (u); f may be called a **partial function** provided every element assigned is uniquely uniquely assigned but this fails at v. It fails to be onto (w) and one-to-one (x). However, if the arrow from v to x is replaced with an arrow from u to w then the mapping becomes a bijective function.



Basic facts: a function $f : [n] \longrightarrow [n]$ (i.e. when $m = n$) is a bijective if and only if it is one-to-one if and only if it is onto. In this case f defines a permutation of $[n]$. Conversely, if $f : [m] \longrightarrow [n]$ is a bijection then we must have $m = n$.

We are interested in counting how many possible mappings $f : [m] \longrightarrow$ with the above properties there can be.

Functions Each entry in $[m]$ may be mapped to any of the n entries in $[n]$. So by the Product Principle there are $n \times n \times \ldots \times n = n^m$ functions.

One-to-one functions Now each entry in $[m]$ may only be mapped to entries in $[n]$ not already 'taken'. So $n \times n \times \ldots \times n$ is replaced with $n \times n - 1 \times \ldots \times (n - k + 1)$:

$$\begin{cases} 0 & \text{if } n < m \quad \text{equivalent to Pigeonhole Principle} \\ \dfrac{n!}{(n-m)!} & \text{if } m \leq n \end{cases}$$

Onto functions There is a converse zero count; the other case is quite a bit more complicated:

$$\begin{cases} 0 & \text{if } m < n \\ \displaystyle\sum_{k=0}^{n-1} (-1)^k \binom{n}{k} (n-k)^m & \text{if } n \leq m \end{cases}$$

Proof: $\boldsymbol{m < n}$. Suppose, for a contradiction, that $f : [m] \longrightarrow [n]$ is an onto function. Put each $i \in [n]$ into a box labelled with the least element of $[m]$ which maps to i. By the Pigeonhole Principle some box receives more than one element of $[n]$ but then the corresponding element of $[m]$ is mapped non-uniquely to $[n]$ contradicting the fact that f is a function.

$\boldsymbol{n \leq m}$. Let A_i be the set of functions from $[m]$ to $[n]$ which *fail* to map to i. Then $|A_1 \cup A_2 \cup \ldots \cup A_n|$ counts the number of functions which fail to be surjections. Since the total number of functions from $[m]$ to $[n]$ is n^m we have

$$\#\text{surjections} = n^m - |A_1 \cup A_2 \cup \ldots \cup A_n|.$$

Now the number of functions which do not map to $X \subset [n]$ is $\left|\bigcap_{j\in X} A_j\right|$. This is the same as the number of functions $[m] \longrightarrow [n - |X|]$, so $\left|\bigcap_{j\in X} A_j\right| = (n - |X|)^m$.

Now we apply Inclusion-Exclusion:

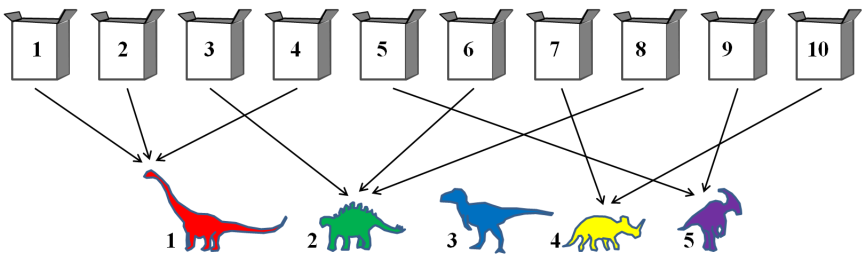
$$\begin{aligned}
\left|\bigcup_{i=1}^{n} A_i\right| &= \sum_{k=1}^{n}\left((-1)^{k+1}\sum_{\substack{X\subseteq[n]\\|X|=k}}\left|\bigcap_{j\in X}A_j\right|\right) = \sum_{k=1}^{n}\left((-1)^{k+1}\sum_{\substack{X\subseteq[n]\\|X|=k}}(n-|X|)^m\right)\\
&= \sum_{k=1}^{n}(-1)^{k+1}\binom{n}{k}(n-k)^m = \sum_{k=1}^{n-1}(-1)^{k+1}\binom{n}{k}(n-k)^m \text{ (since the summand is zero for } k = n).
\end{aligned}$$

$$\begin{aligned}
\text{So: \#surjections} &= n^m - |A_1 \cup A_2 \cup \ldots \cup A_n| = n^m - \sum_{k=1}^{n-1}(-1)^{k+1}\binom{n}{k}(n-k)^m\\
&= \binom{n}{0}(n-0)^m + \sum_{k=1}^{n-1}(-1)^{k}\binom{n}{k}(n-k)^m = \sum_{k=0}^{n-1}(-1)^{k}\binom{n}{k}(n-k)^m.
\end{aligned}$$

Bijections by the Basic Facts given above, the number of bijections is given by

$$\begin{cases} 0 & \text{if } m \neq n \\ n! & \text{if } n \leq m \end{cases}$$

Example. The Coupon Collectors Problem Suppose we have n different dinosaur models to collect from cereal packets each containing one dinosaur selected uniformly at random from n. What is the probability that we will have all n dinosaurs after munching our way through m packets of cerals?



Suppose we take $n = 5$ and $m = 10$ for the sake of a concrete example. We are mapping 10 packets to 5 dinosaurs and we want a surjection. The number of possible surjections is $\sum_{k=0}^{4}\binom{5}{k}(5-k)^{10} \approx 5.1 \times 10^6$. The total number of functions is $5^{10} \approx 9.8 \times 10^6$. So the probability of a surjection is about 5.1/9.8, about a 52% chance.