# Chapter 2. Section 3

## Variations on Permutations and Combinations

**Permutations of distinct objects** we can arrange $k$ out of $n$ distinct objects in a line (i.e. **linear arrangements**) in

$$n(n-1)(n-2)\cdots(n-k+1)=\frac{n!}{(n-k)!}$$

ways.

**Permutations of non-distinct objects** if our $n$ objects consist of $c_1$ copies of object 1, $c_2$ copies of object 2, $\ldots c_N$ copies of object $N$, where

$$c_1+c_2+\ldots+c_N=n,\ N\leq n,$$

then the number of linear arrangements of all $n$ objects is

$$\frac{n!}{c_1!c_2!\cdots c_N!}=\frac{n!}{\prod c_i!}.$$

This formula can be derived by **labelling** objects. *It is usually easier to count using labelled objects than unlabelled (identical) objects!*

**Example** How many linear arrangements can be made from the collection of objects $p, p, q, q, q$?

Label the objects to make them different: $p_1, p_2, q_1, q_2, q_3$. Now there are $5! = 120$ linear arrangements. We might take the first six as:

$$\begin{array}{ccccc}
p_1 & p_2 & q_1 & q_2 & q_3\\
p_1 & p_2 & q_1 & q_3 & q_2\\
p_1 & p_2 & q_2 & q_1 & q_3\\
p_1 & p_2 & q_2 & q_3 & q_1\\
p_1 & p_2 & q_3 & q_1 & q_2\\
p_1 & p_2 & q_3 & q_2 & q_1
\end{array}$$

But these are all the same arrangement of two $p$s and three $q$s: it has been repeated $3! = 6$ times for all the ways we can arrange $q_1, q_2, q_3$. And then it will repeat another six times:

$$\begin{array}{ccccc}
p_2 & p_1 & q_1 & q_2 & q_3\\
p_2 & p_1 & q_1 & q_3 & q_2\\
 & & \ldots & &
\end{array}$$

So our list of six repetitions is repeated $2! = 2$ times for the two arrangements of $p_1, p_2$.

We see that every linear arrangement of $p, p, q, q, q$ will be repeated $2!3!$ times for the different orderings of the $p_i$ and $q_i$.

So the answer is $\dfrac{5!}{2!3!}=\dfrac{120}{2\times 6}=10.$

Dividing by $n!$ **removes the labelling** from arrangements of $n$ labelled objects.

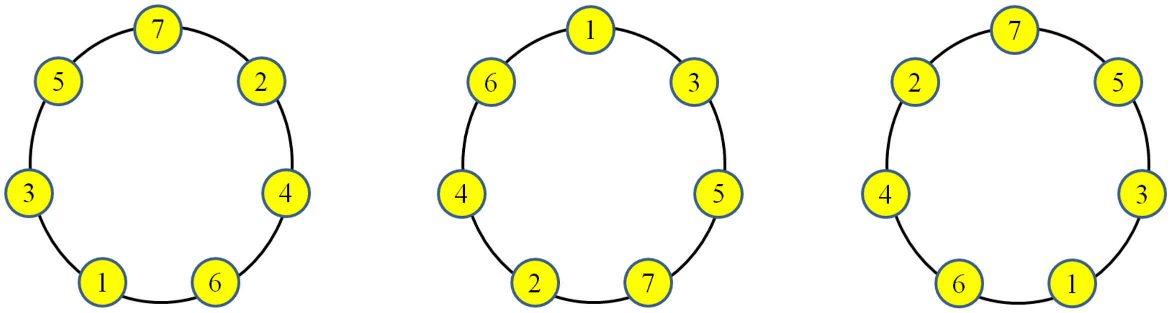
**Example** How many linear arrangements are there of **three** objects chosen from $\{p, p, q, q, q\}$?

We cannot avoid checking each possible subset choice: $\{p, p, q\}$, $\{p, q, q\}$, $\{q, q, q\}$ and applying the formula separately:

$$\#\text{arragements}=\frac{3!}{2!1!}+\frac{3!}{1!2!}+\frac{3!}{0!3!}=\frac{6}{2}+\frac{6}{2}+\frac{6}{6}=7.$$

**Cyclic permutations of distinct objects** Two arrangement of $n$ objects in a cycle are considered the same if one is a cyclic permutation of the other.

**Example** Here are three cyclic arrangements of $[7] = \{1, 2, \ldots, 7\}$ (**note the abbreviation** $[n] = \{1, 2, \ldots, n\}$).




The second is a cyclic permutation of the first obtained by cycling clockwise four positions. The third is considered to be a different arrangement.

There are $(n-1)!$ cyclic arrangements of $n$ distinct objects, obtained by fixing the first object (at the top of the cycle) and then permutating the remaining $n-1$ objects relative to this fixed object.

If **mirror image** arrangements are also considered identical then the third arrangement above is also considered to be the same as the first. This may be thought of as asking how many **necklaces** can be made from $n$ differently coloured beads. All three arrangements above would count as the same necklace—they are just being worn differently.

The number of arrangements of $n$ distinct objects considered identical under cyclic permutations and mirror-image reflections is $(n-1)!/2$.

**Combinations chosen without replacement** We can choose a subset of $k$ objects from a set of $n$ distinct objects in $\binom{n}{k} = \dfrac{n!}{(n-k)!k!}$ ways.

**Combinations chosen with replacement** Imagine drawing $k$ objects from a bag of $n$ distinct objects but after recording each choice we replace it in the bag.

**Example:** Our bag contains $[6] = \{1, 2, \ldots, 6\}$. We draw out five, replacing after each draw. The result could be: 1, 5, 1, 5, 3.

The number of ways of choosing $k$ objects from $n$ with replacement is $\binom{n+k-1}{k}$.

**Proof:** Suppose our choices are arranged in non-decreasing order as:

$$
\begin{array}{lccccccc}
 & c_1 & \le & c_2 & \le & & \ldots \le & c_k. \\
\text{Add:} & 0 & & 1 & & 2 & \ldots & k-1 \\
\text{to get:} & d_1 & < & d_2 & < & & \ldots < & d_k.
\end{array}
$$

The $d_i$ lie in the range $1, \ldots, n + (k-1)$ and they are all different because the $\le$'s have become $<$'s.

The possible choices of $k$ values for $c_i$ from $1, \ldots, n$ are in one-to-one (bijective) correspondence with the possible choices of $k$ values for $d_i$ from $1, \ldots, n-k+1$, because, given a set of distinct $d_i$ values we can map back to an 'original' set of $c_i$ values by subtracting $0, 1, \ldots, n-1$. There are $\binom{n+k-1}{k}$ choices of $d_i$ values and this is equal to the number of $c_i$ choices.

**Example:** Taking our earlier sequence of $c_i$'s in non-decreasing order:

$$\begin{array}{lccccccccc} & 1 & \le & 1 & \le & 3 & \le & 5 & \le & 5. \\ \text{Add:} & 0 & & 1 & & 2 & & 3 & & 4 \\ \text{to get:} & 1 & < & 2 & < & 5 & < & 8 & < & 9 \end{array}$$

producing a subset $\{1, 2, 5, 8, 9\}$ chosen from $\{1, 2, \ldots, 10\}$.

Going the other way, starting with, say, $\{1, 2, 8, 9, 10\}$ chosen from $\{1, 2, \ldots, 10\}$:

$$\begin{array}{lccccccccc} & 1 & < & 2 & < & 8 & < & 9 & < & 10. \\ \text{Subtract:} & 0 & & 1 & & 2 & & 3 & & 4 \\ \text{to get:} & 1 & \le & 1 & \le & 6 & \le & 6 & \le & 6 \end{array}$$

producing a sequence chosen with replacement from $\{1, 2, \ldots, 6\}$.

**Distribution** We have $n$ **identical** objects which we want to place into $k$ boxes. How many ways may this be done?

**Example:** We have four (identical) sweets to distribute to three children. How many ways can this be done?

On the right we show a systematic way to list the choices, which also shows how to derive the formula in general. We think of each possible distribution as a nonnegative integer solution to the equation

$$c_1 + c_2 + c_3 = 4.$$

There are 4 units to distribute between the variables and $3 - 1$ '+' signs which carry out the distribution. Make a row of $4 + 3 - 1 = 6$ boxes and chose which 2 boxes get the +'s. Now you can read off the corresponding solutions to the equation: the first two has zero boxes before the first +, so $c_1 = 0$; zero more boxes before the second +, so $c_2 = 0$, leaving $c_3 = 4$, etc.

So the answer to our question is: how many ways can we choose $3 - 1$ boxes from $4 + 3 - 1$: that is, $\binom{6}{2} = 15$.

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $\boxplus$ | $\boxplus$ | $\square$ | $\square$ | $\square$ | $\square$ | 0 | 0 | 4 |
| $\boxplus$ | $\square$ | $\boxplus$ | $\square$ | $\square$ | $\square$ | 0 | 1 | 3 |
| $\boxplus$ | $\square$ | $\square$ | $\boxplus$ | $\square$ | $\square$ | 0 | 2 | 2 |
| $\boxplus$ | $\square$ | $\square$ | $\square$ | $\boxplus$ | $\square$ | 0 | 3 | 1 |
| $\boxplus$ | $\square$ | $\square$ | $\square$ | $\square$ | $\boxplus$ | 0 | 4 | 0 |
| $\square$ | $\boxplus$ | $\boxplus$ | $\square$ | $\square$ | $\square$ | 1 | 0 | 3 |
| $\square$ | $\boxplus$ | $\square$ | $\boxplus$ | $\square$ | $\square$ | 1 | 1 | 2 |
| $\square$ | $\boxplus$ | $\square$ | $\square$ | $\boxplus$ | $\square$ | 1 | 2 | 1 |
| $\square$ | $\boxplus$ | $\square$ | $\square$ | $\square$ | $\boxplus$ | 1 | 3 | 0 |
| $\square$ | $\square$ | $\boxplus$ | $\boxplus$ | $\square$ | $\square$ | 2 | 0 | 2 |
| $\square$ | $\square$ | $\boxplus$ | $\square$ | $\boxplus$ | $\square$ | 2 | 1 | 1 |
| $\square$ | $\square$ | $\boxplus$ | $\square$ | $\square$ | $\boxplus$ | 2 | 2 | 0 |
| $\square$ | $\square$ | $\square$ | $\boxplus$ | $\boxplus$ | $\square$ | 3 | 0 | 1 |
| $\square$ | $\square$ | $\square$ | $\boxplus$ | $\square$ | $\boxplus$ | 3 | 1 | 0 |
| $\square$ | $\square$ | $\square$ | $\square$ | $\boxplus$ | $\boxplus$ | 4 | 0 | 0 |

In general, the answer to the distribution problem of putting $n$ identical objects into $k$ boxes is:

$$\binom{n+k-1}{k-1}.$$

Pascal's Triangle is symmetrical: for any $n$ and $k$, $\binom{n}{k} = \binom{n}{n-k}$. So we have

$$\binom{n+k-1}{n+k-1-(k-1)} = \binom{n+k-1}{n},$$

and we see that

> The number of ways of distributing $n$ identical objects into $k$ boxes equals the number of ways to choose $n$ objects from $k$ distinct objects with replacement.

You can try and find a **bijective proof** of this fact!