Chapter 2. Section 2

Example (a typical exam-type question)

(a) State the Product Principle and the Summation Principle of combinatorial counting.

(b) Two 6-sided dice are thrown. By using the Product and Summation Principles, as appropriate, determine the following:

(i) How many outcomes are there?

(ii) How many possible outcomes are there for the sum on the two dice?

(iii) What is the probability that the two dice sum to 6?

(iv) What is the expected sum on the two dice?

(c) Two players A and B each throw two dice. Let P_{AB} denote the probability that the sum of A's two dice is equal to the sum of B's two dice.

(i) What is the value of P_{AB}?

(ii) If an n-sided dice is used, give a formula in n for the value of P_{AB}.

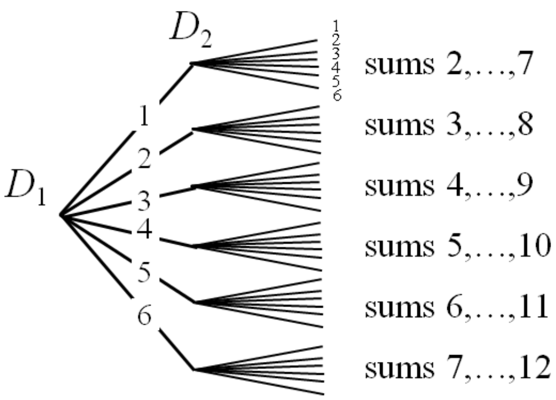
Solution:

(a) These are given in the lecture notes (Week 1, lecture 3). For full marks, textbook definitions should be given as exactly as possible!

(b) When two dice are thrown,

(i) There are $6 \times 6 = 36$ outcomes, by the Product Principle.

(ii) A tree is not essential, unless the question requires it, but can be very helpful!



Nor does a drawing have to be complete. The above is sufficient to reveal that there are eleven possible sum outcomes: $2, \ldots, 12$.

Note that it is **incorrect** to say that the Summation Principle is used here: if the two dice are regarded as two trials then their outcomes are not exclusive!

(iii) If you didn't need the tree in part (ii) it may become useful now! There are five outcomes from D_1 which lead to a dice sum of 6. For each of these, the sum 6 occurs in only one way. So the required probability is #ways to get 6/#outcomes = $5/36$.

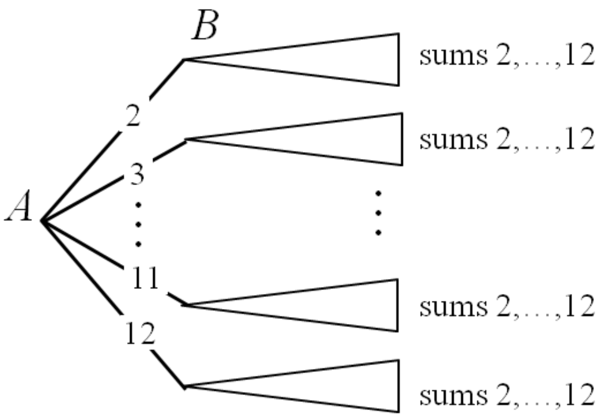
(iv) Expected sum $\mathbb{E}$ is given as $\mathbb{E} = \sum_{\text{sum}=k} p_k \times k$ where p_k is the probability that the sum is k. For example $p_6 = 5/36$. We get

$$\begin{aligned}
\mathbb{E} &= \frac{1}{36}\times 2 + \frac{2}{36}\times 3 + \frac{3}{36}\times 4 + \ldots + \frac{3}{36}\times 10 + \frac{2}{36}\times 11 + \frac{1}{36}\times 12 \\
&= \frac{1}{36}(2+12) + \frac{2}{36}(3+11) + \ldots + \frac{5}{36}(6+8) + \frac{6}{36}\times 7 \\
&= \frac{1}{36}\Big(14\times(1+2+3+4+5+6) - 7\times 6\Big) \\
&= \frac{1}{36}\left(14\times\frac{1}{2}\times 6\times 7 - 7\times 6\right) = \frac{(7-1)\times 6\times 7}{36} = 7.
\end{aligned}$$

So $\mathbb{E}$ which is 'mean sum' is 7, the same as the median sum (which makes sense since the tree in part (ii) was symmetrical, in terms of sum outcomes).

(c) To establish P_{AB}, the probability that the sum of A's two dice is equal to the sum of B's two dice, we need a **new tree**!

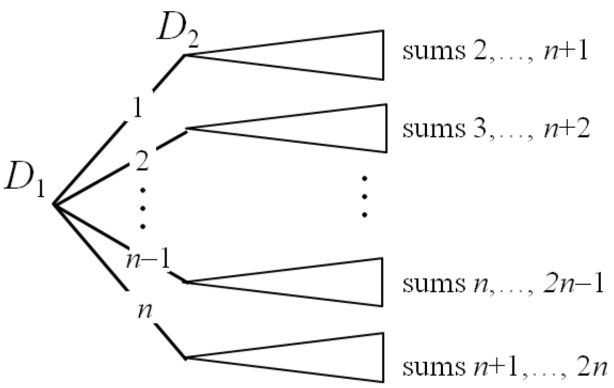
(i) The value of P_{AB} is the sum, over all endpoints in the tree corresponding to equal dice sums, of the probabilities of those outcomes.

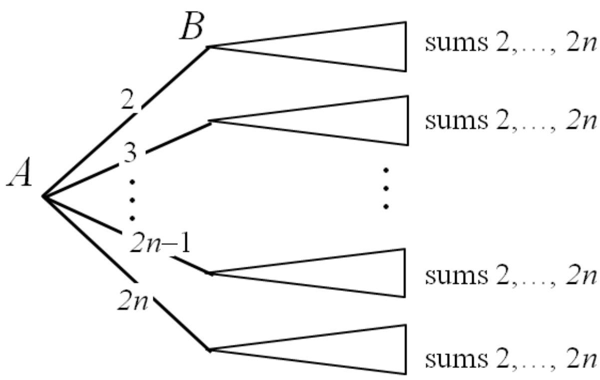


There are eleven equal-sum outcomes: 2 2; 3 3 ...; 11 11 and 12 12. But they have different probabilities because a dice sum of 2 occurs with probability 1/36 while a dice sum of 6 occurs with probability 5/36, etc:

$$\begin{aligned}
\text{Prob } A \text{ sum equals } B \text{ sum} &= \left(\frac{1}{36}\right)^2 + \left(\frac{2}{36}\right)^2 + \ldots + \left(\frac{6}{36}\right)^2 + \ldots + \left(\frac{2}{36}\right)^2 + \left(\frac{1}{36}\right)^2 \\
&= \frac{1}{36^2}\left(2\left(1^2+2^2+\ldots+6^2\right) - 6^2\right) = \frac{146}{1296} \approx 0.11 \text{ or } 11\%.
\end{aligned}$$

(ii) For n-sided dice we just need to determine how our trees generalise from $n = 6$:



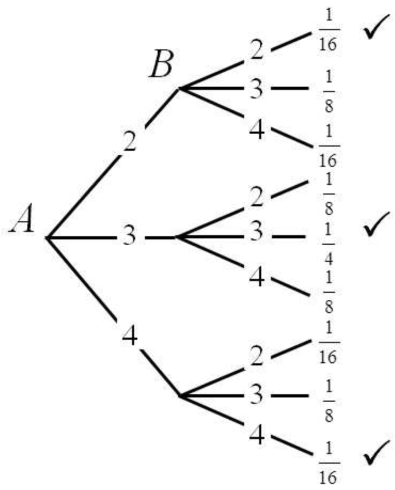


The summation has just the same 'shape' as before:

$$\begin{aligned}\text{Prob } A \text{ sum equals } B \text{ sum} &= \left(\frac{1}{n^2}\right)^2 + \left(\frac{2}{n^2}\right)^2 + \ldots + \left(\frac{n}{n^2}\right)^2 + \ldots + \left(\frac{2}{n^2}\right)^2 + \left(\frac{1}{n^2}\right)^2 \\ &= \frac{1}{n^4}\left(2\left(1^2 + 2^2 + \ldots + n^2\right) - n^2\right) \\ &= \frac{1}{n^4}\left(2.\frac{1}{6}n(n+1)(2n+1) - n^2\right) = \frac{2n^2+1}{3n^3}.\end{aligned}$$

We can check this is consistent with our answer for $n = 6$, when $\frac{2n^2+1}{3n^3} = \frac{73}{648} \approx 0.11$.

Better still we can check for $n = 2$ by actually listing all the probabilities, because there are not too many. This is not part of our exam question answer, of course, but it is a very good way of making sure you really know what is going on!



There: probability of getting same sum is $1/16 + 1/4 + 1/16 = 6/16 = 3/8$ which is the same as $\frac{2 \times 2^2 + 1}{3 \times 2^3} = \frac{9}{24}$.