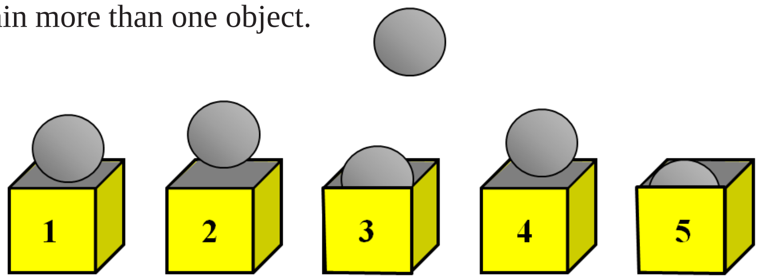
Chapter 2. Section 1

Proof Principles

The Pigeon Hole Principle: If n objects are placed into $n-1$ boxes then at least one box must contain more than one object.



OBVIOUS! You will say. But you do see this proved in some text books. And a generalised version is proved in the Week 1 Coursework sheet.

Example 1 There are n people at a party. Prove that at least two people know exactly the same number of people at the party.

We put each person into a box labelled with the number of people they know at the party: 0, 1, 2, ... up to $n-1$.

Either box 0 or box $n-1$ must be empty because if somebody knows zero people then nobody else can know everyone else ($n-1$) and vice versa (0 and $n-1$ are exclusive).

So we are actually putting n people into $n-1$ boxes (although we don't know if these are boxes $0, 1, \ldots, n-2$ or boxes $1, 2, \ldots, n-1$).

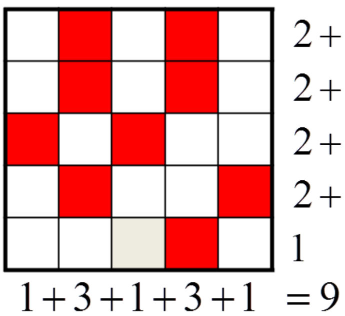
So at least one box has more than one person: there are two people who know the same number of people at the party.

Example 2 Among any $n+1$ values chosen from $1, 2, 3, \ldots, 2n$ there is a pair of values one of which is a multiple of the other.

Proof: Write each of the values uniquely in the form $2^a \times b$ for some $a \geq 0$ and some odd number b, $1 \leq b \leq 2n-1$. There are n possible values for b in this range. So by the Pigeon Hole Principle two of the $n+1$ chosen values must share the same b value. Suppose, in ascending order, they are $2^a b < 2^{a'} b$. Then 2^a divides $2^{a'}$ so the first value divides the second.

We say that this result is **best possible** meaning that it is no longer guaranteed to be true if we only select n values. Because among the n values $n+1, n+2, \ldots, 2n$ no pair has one dividing the other.

Double Counting: this is a rather general idea about how to show an equality is true. It may be illustrated as below:



The idea is to find a formula F_r for how many coloured squares there are by counting along the rows and another formula F_c for counting down the columns. Since the total number of coloured squares is the same in each case we must have an equality $F_r = F_c$. But the picture just suggests how things work, there may not actually be rows and columns involved!

Example 1 Let $S(n)$ denote the sum of the first n positive integers: $1+2+\ldots+n-1+n$. Prove that $S(n) = \frac{1}{2}n(n+1)$.

How many dots appear in the following picture?

Counting one way, the top triangle is $S(4)$; the bottom triangle is also $S(4)$, and the row in between is 5. This gives a total of $S(4)+5+S(4)$. Counting another way the complete picture is a square containing 5^2 dots.

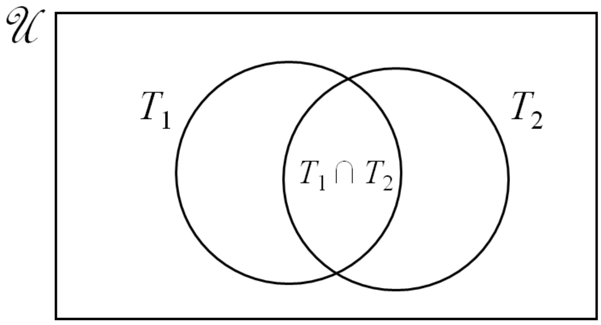
In general, then, $S(n)+n+1+S(n) = (n+1)^2$. Rearranging, $2S(n) = (n+1)((n+1)-1)$ and the formula for $S(n)$ follows.

Example 2 Here is another proof of the defining equation $\binom{n}{k} = \frac{n!}{k!}(n-k)!$.

We count how many ordered sequences of k objects can be formed from $n \geq k$ distinct objects. Firstly choose k objects in $\binom{n}{k}$ ways. Then rearrange these in all possible ways to get a total of $\binom{n}{k}k!$. Counting another way, we saw previously that there are $n \times n-1 \times n-2 \times \ldots\ldots n-k+1 = \frac{n!}{(n-k)!}$ ordered sequences of k objects. Both counts are the same, so $\binom{n}{k}k! = \frac{n!}{(n-k)!}$, and the result follows.

Note: this is quite different from the 'double counting' which means including something twice when you are counting something. This is usually a problem and it is dealt with specifically by the following technique.

Inclusion-Exclusion: If the outcomes of two trials T_1 and T_2 are exclusive then the Summation Principle says that choosing between T_1 and T_2 gives a total of $\#T_1 + \#T_2$ outcomes. But if there are **not** exclusive then this double-counts the outcomes in the overlap:



We can correct for this overcounting provided we can count $\#T_1 \cap T_2$. Then

$$\#T_1 \cup T_2 = \#T_1 + \#T_2 - \#T_1 \cap T_2.$$

Example 1 Roughly how many numbers in the range $1, \dots, 100$ are divisible by either 2 or 3 or both?

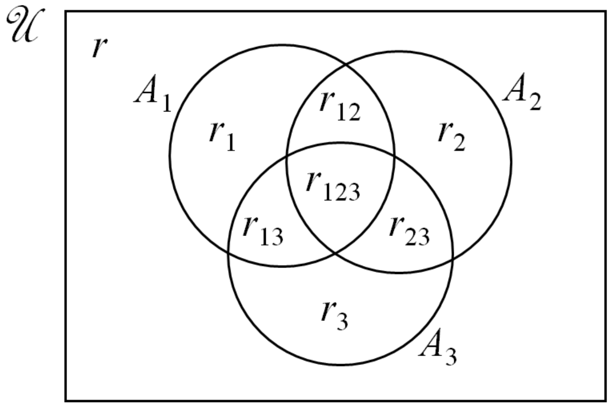
There are 50 even numbers in the range; there are roughly 33 multiples of 3; about half (16) of these are also even numbers.

So divisible by 2 or 3 (or both) $\approx 50 + 33 - 16 = 67$ (which is actually exactly accurate!)

The overlap calculation for two trials extends to give a three-way **inclusion-exclusion formula** which is usually specified in terms of set cardinalities as follows:

$$\begin{aligned} |A_1 \cup A_2 \cup A_3| \;=\; & |A_1| + |A_2| + |A_3| \\ & -\big(|A_1 \cap A_2| + |A_1 \cap A_3| + |A_2 \cap A_3|\big) \\ & +|A_1 \cap A_2 \cap A_3| \end{aligned} \tag{1}$$

You can check each of the regions in the picture below to see that its final inclusion count is 1:



For example, r_{12} is included twice via $|A_1| + |A_2|$ and removed once via $-|A_1 \cap A_2|$.

Example 2 Roughly how many primes are there among the 48 numbers in the range $1 \dots 48$?

This is the same as Example 1 except this time we count for division by 2, 3 and 5 (any composite number < 49 will divide by one of these three).

There are roughly:

24 even numbers;
16 multiples of 3 with 8 being even;
9 multiples of 5 with 4 being even and 3 being multiples of 3 of which 1 will be even.

So in the range $1 \dots 48$ there are roughly $24 + 16 + 9 - 8 - 4 - 3 + 1 = 35$ numbers divisible by 2, 3 or 5. This includes the three primes themselves, so there are roughly $48 - 35 + 3 = 16$ primes (the exact number is 15).

What if we have n subsets of $\mathscr{U}$ the universal set? Using $[n]$ to denote $\{1, 2, \dots, n\}$, Inclusion-Exclusion generalises to give:

$$\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{k=1}^{n}\left((-1)^{k+1} \sum_{\substack{X \subseteq [n] \\ |X|=k}} \left|\bigcap_{j \in X} A_j\right|\right) \quad \text{(matches eqn (1) above when } n = 3\text{)}$$

$$= \sum_{\substack{X \subseteq [n] \\ X \neq \emptyset}} (-1)^{|X|+1} \left|\bigcap_{j \in X} A_j\right| \quad \text{(less to remember but more `condensed').}$$