Chapter 10. Section 3

Counting Perfect Matchings in Planar Graphs

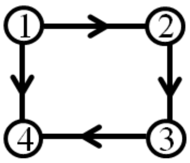
We do not know how to count perfect matchings in bipartite graphs, because we do not have a better way to do this than to calculate the permanent of the incidence matrix, and calculating permanents is hard. However, we can count perfect matchings in planar graphs. And we already have all the techniques at our disposal!

In planar graphs, perfect matchings are often called **1-factors**: they are spanning subgraphs in which every vertex has degree exactly 1. However, we will keep to the name matching: even though we are no longer matching ground set elements to $\mathcal{A}$-sets we are still matching up graph vertices in pairs: a perfect matching in an n-vertex graph is a set of $n/2$ edges each matching up a different pair of adjacent vertices. Notice that, of course, n must be even for this to happen!

We can count perfect matchings in planar graphs thanks to a wonderful 1967 result of the Dutch physicist Pieter Kasteleyn. Recall that the $n \times n$ **adjacency matrix** of an n-vertex simple graph has a ‘1’ in row i, column j if and only if vertex i is adjacent to vertex j. If the edges of the graph are **oriented** (given a direction) then we have a **signed adjacency matrix**:

$$\text{entry } ij = \begin{cases} +1 & \text{if vertex } i \text{ has an edge directed towards vertex } j \\ -1 & \text{if vertex } j \text{ has an edge directed towards vertex } i \\ 0 & \text{otherwise.} \end{cases}$$

An example is shown on the right. You might like to take a moment to convince yourself that the graph has 2 perfect matchings; that the signed adjacency matrix has determinant 4; and that the permanent of this matrix has the same value as the determinant.



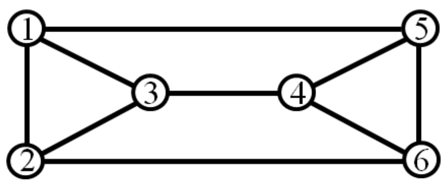
$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & 1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ -1 & 0 & -1 & 0 \end{pmatrix} \end{array}$$

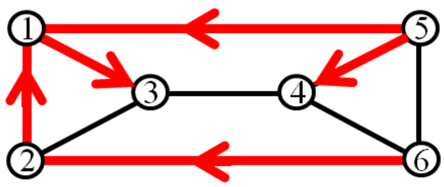
Theorem (Kastelyn’s Theorem) Suppose that G is a planar graph drawn in the plane. Then

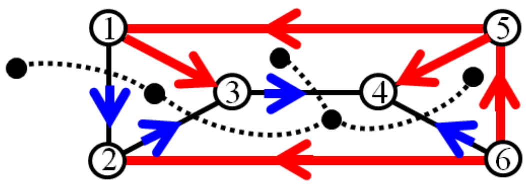
1. we can orient the edges so that every face has an odd number of clockwise-oriented edges, and
2. if $A(G)$ is the signed adjacency matrix of the orientation of G then
$$\#\text{perfect matchings of } G = \sqrt{\det(A(G))}.$$

The example of the right illustrates how part (1) of Kastelyn’s theorem is proved.

1. Take a spanning tree T of G and orient its edges arbitrarily (the heavy, red edges in the 2nd drawing on the right).
2. Take a spanning tree T^* in the dual graph whose edges cross edges of G that are not in T (the dotted edges in the bottom drawing).
3. Starting at leaves of T^* located in internal faces orient the G edges crossed by T^* edges so as to give each face an odd number of clockwise-oriented edges (the heavy, blue arrows in the bottom drawing).

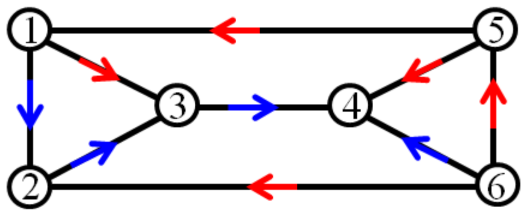






The faces of G, in order of orientation, receive the following numbers of clockwise edges: 1345:**1**; 456:**1**; 2346:**3**; 123:**1**. As a check, the outside edge should also be ‘odd’: 1256:**1**;.

Our example has resulted in an oriented version of the original graph G. It is shown below together with its signed adjacency matrix:



$$A(G) = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{pmatrix} & 1 & 1 & & -1 & \\ -1 & & 1 & & & -1 \\ -1 & -1 & & 1 & & \\ & & -1 & & -1 & -1 \\ 1 & & & 1 & & -1 \\ & 1 & & 1 & 1 & \end{pmatrix} \end{array}$$

We can evaluate $\det A(G)$ by a mixture of Gaussian elimination and expanding on 'pivot' elements (unique nonzero entries in a column):

$$\det \begin{pmatrix} & 1 & 1 & & -1 & \\ -1 & & 1 & & & -1 \\ -1 & -1 & & 1 & & \\ & & -1 & & -1 & -1 \\ 1 & & & 1 & & -1 \\ & 1 & & 1 & 1 & \end{pmatrix} \begin{array}{l} \\ 2 \to 2+5 \\ 3 \to 3+5 \\ \\ \\ \\ \end{array} = \det \begin{pmatrix} & 1 & 1 & & -1 & \\ & & 1 & 1 & & -2 \\ & -1 & & 2 & & -1 \\ & & -1 & & -1 & -1 \\ 1 & & & 1 & & -1 \\ & 1 & & 1 & 1 & \end{pmatrix}$$

$$= +1 \times \det \begin{pmatrix} 1 & 1 & & -1 & \\ & 1 & 1 & & -2 \\ -1 & & 2 & & -1 \\ & -1 & & -1 & -1 \\ 1 & & 1 & 1 & \end{pmatrix} \begin{array}{l} \\ \\ 3 \to 3+1 \\ \\ 5 \to 5-1 \end{array} = \det \begin{pmatrix} 1 & 1 & & -1 & \\ & 1 & 1 & & -2 \\ & 1 & 2 & -1 & -1 \\ & -1 & & -1 & -1 \\ & -1 & 1 & 2 & \end{pmatrix}$$

$$= +1 \times \det \begin{pmatrix} 1 & 1 & & -2 \\ 1 & 2 & -1 & -1 \\ -1 & & -1 & -1 \\ -1 & 1 & 2 & \end{pmatrix} \begin{array}{l} \\ 2 \to 2-1 \\ 3 \to 3+1 \\ 4 \to 4+1 \end{array} = \det \begin{pmatrix} 1 & 1 & & -2 \\ & 1 & -1 & 1 \\ & 1 & -1 & -3 \\ & 2 & 2 & -2 \end{pmatrix}$$

$$= +1 \times \det \begin{pmatrix} 1 & -1 & 1 \\ 1 & -1 & -3 \\ 2 & 2 & -2 \end{pmatrix}$$

$$= 2 \times \det \begin{pmatrix} 1 & -1 & 1 \\ 1 & -1 & -3 \\ 1 & 1 & -1 \end{pmatrix}$$

$$= 2 \times \det \begin{pmatrix} 1 & -1 & 1 \\ & & -4 \\ & 2 & -2 \end{pmatrix}$$

$$= 16.$$

And we have #perfect matchings in $G = \sqrt{\det A(G)} = \sqrt{16} = 4$.

For such a small graph it is not hard to confirm this combinatorially; the perfect matchings are:

12	34	56;
13	26	45;
15	23	46;
15	26	34.