# Chapter 9. Section 3

### Set Systems

Take a finite set $E$, called the **ground set**. A **set system** on $E$ is a collection $\mathcal{A} = \{A_1, A_2, \ldots, A_t\}$ of subsets of $E$. Although set notation is used for convenience, subsets of $\mathcal{A}$ may be repeated.

**Examples** (1) $E = \{a, b, c, d, e\}$, $\mathcal{A} = \{\{a, c\}, \{a, d\}, \{a\}, \{a\}, \{d, e\}\}$.

(2) A graph $G$ may be considered as a set system: the vertex set $V$ is the ground set and the edge set is a collection of subsets of $V$ (which may be repeated, in the case of multiple edges).[1]

(3) Suppose $E$ is the set of students at a university and $\mathcal{A}$ is the collection of student clubs and societies. It may certainly be the case that some clubs comprise exactly the same subset of students (although they will presumably do different things when meeting together as different clubs).

(4) The set of all subsets of $[n]$ is a set system in which all sets are distinct. It has size $2^n$.

(5) A set of distinct subsets of $[n]$, all of which have size $k$, for some $k \leq n/2$, and in which no pair is disjoint, is called an **intersecting** family. It can be proved that it has size at most $\binom{n-1}{k-1}$ (this is the famous Erdős-K0-Rado Theorem).

The property of set systems that we will focus on is the following: can we find a subset of the ground set $E$ for which each element may 'choose' a **different** subset in $\mathcal{A}$ to which it belongs. Such a subset is called a **system of distinct representatives** or a **transversal**. We shall use the latter term because it is shorter. But certainly our subset elements may be thought of as 'representing' the $\mathcal{A}$ sets which they choose. In example (3), we might think of a transversal as being an election of presidents for the clubs and societies: the presidents must belong to the clubs they represent; and different clubs should have distinct presidents, to avoid conflict of interests.

A **complete transversal** is one in which **every** set in $\mathcal{A}$ is represented. To have a complete transversal it is clearly necessary that $|\mathcal{A}| \leq |E|$. But this is not sufficient: in example (1) above, $|E| = 5 = |\mathcal{A}|$ but there can be no complete transversal, because the two copies of $\{a\}$ can only be chosen by $a$, and no element of $E$ may chose more than one set. Generalising this example leads to a necessary and sufficient condition for a transversal to exist.

**Hall's Theorem** A set system $\mathcal{A} = \{A_1, A_2, \ldots, A_t\}$ on ground set $E$ has a complete transversal if and only if, for every subset $W$ of $E$, we have

$$|W| \leq \left| \bigcup_{i \in W} A_i \right|. \tag{1}$$

In example (1), $A_3 = A_4 = \{a\}$ and if $W = \{3, 4\}$ then $|\cup_{i \in W} A_i| = |\{a\} \cup \{a\}| = 1$. Since $|W| \not\leq 1$, the condition of Hall's Theorem fails and there can be no complete transversal.

Hall's Theorem is sometimes called the **Marriage Theorem**: think of $E$ as a set of men and $A_i$ as being the men whom woman i will agree to marry. A complete transversal is a matching of each woman to a different acceptable man. This is only possible if any subset $W$ of women collectively like at least $|W|$ men.

Hall's Theorem is an example of what is called in combinatorics a **good characterisation**: it allows both existence and non-existence of something to be **certified**. If a complete transversal exists then we can produce one as a 'certificate' of the fact; if no complete transversal exists then we can produce a set $W$ failing Hall's condition as our certificate of non-existence. We have already seen another example of a good characterisation: Kuratowski's Theorem on graph planarity.

[1] It is confusing that $E$ is **not** the ground set in this case, but in fact $E$ is chosen for denoting ground sets in general because there is a much more important use of set systems based on graphs in which the edge set $E$ *is* the ground set.