Chapter 1. Section 3

Basic Principles

Two Definitions

Factorial: $n! = n \times n-1 \times n-2 \times \ldots \times 2 \times 1$ for $n \in \mathbb{Z}^{>0}$ and $0! = 1$.

Binomial: 'n choose k' $\binom{n}{k} = \dfrac{n!}{k!(n-k)!}$ for $n \geq k \in \mathbb{Z}^{\geq 0}$ and $\binom{n}{k} = 0$ for $n < k$.

Combinatorial Arguments

1. To prove that the 1st n odd positive integers sum to n^2.

   Algebraically: Using the fact that $1 + 2 + \ldots + n = \frac{1}{2}n(n+1)$:

$$\begin{aligned} 1 + 3 + 5 + \ldots + 2n - 1 &= \sum_{k=1}^{n} (2k-1) \\ &= 2\sum_{k=1}^{n} k - \sum_{k=1}^{n} 1 = \cancel{2} \times \frac{1}{\cancel{2}} n(n+1) - n = n^2. \end{aligned}$$

   No words are necessary — we are completely convinced by symbol manipulation! But we learn nothing except that odd numbers sum to squares.

   Combinatorially: The following sequence also convinces:

   1 **1 + 3** **1 + 3 + 5** **1 + 3 + 5 + 7** . . .

   Symbols have been replaced by **structures**. And this argument tells us **why** odd numbers sum to squares: each odd number is structurally equal to the difference between successive squares.

2. Given that

   $n!$ counts ordered arrangements of n distinct objects,

   and

   $\binom{n}{k}$ counts subsets of size k drawn from a set of size n (hence 'n choose k');

   to prove the algebraic definition $\binom{n}{k} = \dfrac{n!}{k!(n-k)!}$.

   Proof: Suppose X is the number of ordered sequences of k objects drawn from n objects. Then $\binom{n}{k} = \dfrac{X}{k!}$ since each subset of k objects will be counted $k!$ times in X, once for each arrangement of the k subset elements.

Now to form an ordered sequence of k objects we have

n	choices for the 1st object, for each of which there remain
$n-1$	choices for the 2nd object, for each of which there remain
$n-2$	choices for the 3rd object, for each of which there remain
$\ldots$	
$n-k+1$	choices for the kth and final object.

So the total number of choices is

$$X = n \times n-1 \times \ldots \times n-k+1 = \frac{n \times n-1 \times \ldots \times n-k+1 \times \cancel{n-k} \times \cancel{n-k-1} \times \ldots \times \cancel{2} \times \cancel{1}}{\cancel{n-k} \times \cancel{n-k-1} \times \ldots \times \cancel{2} \times \cancel{1}} = \frac{n!}{(n-k)!}.$$

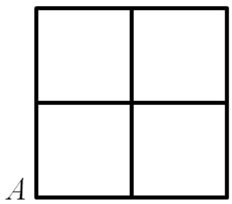
So $\binom{n}{k} = \frac{X}{k!} = \frac{n!}{k!(n-k)!}$, as required.

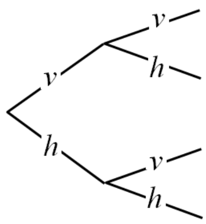
Two Counting Principles

Product Principle: If trial T_1 has m outcomes and trial T_2 has n outcomes *which are independent of the outcome of* T_1, then the total number of outcomes when T_1 followed by T_2 is $m \times n$.

Example: How many walks of length two are there, starting from A in the grid, below left? A walk consists of horizontal or vertical steps from one grid point to the next, visiting any grid point at most once.

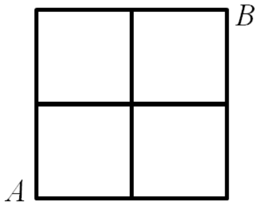
The answer: T_1 = first step, either vertical (v) or horizontal (h). T_2 = second step, v or h; so total walks (outcomes) = $2 \times 2 = 4$. This may be depicted as a tree, as shown below right.

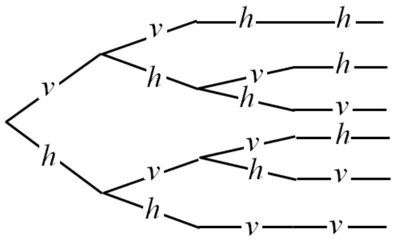




Why do the T_i have to be independent? Suppose we asked how many walks go form A to B in the grid (see below left). It takes four steps to get from A to B, so that is four trials: T_1, T_2, T_3, T_4. But the answer is not $2 \times 2 \times 2 \times 2$, because T_3 and T_4 depend on T_1 and T_2. If the outcome of T_1 and T_2 are both v then T_3 and T_4 are forced to be h.

We can still use a tree and this helps to work though all the possible outcomes: see below right.





Summation Principle: If trial T_1 has m outcomes and trial T_2 has n outcomes, and the outcomes of T_1 and T_2 *are exclusive of each other*, then the total number of outcomes when either T_1 or T_2 is chosen is $m + n$.

Example: Prove Pascal's Rule: for $n, k \geq 1$, $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$.

Proof: recall that $\binom{n}{k}$ counts the number of ways to choose a subset, say U, of size k from a set, say X, of size n.

Let u be any one element of X. Let $T_1 = U$ is chosen to contain u; $T_2 = U$ is chosen to not contain u.

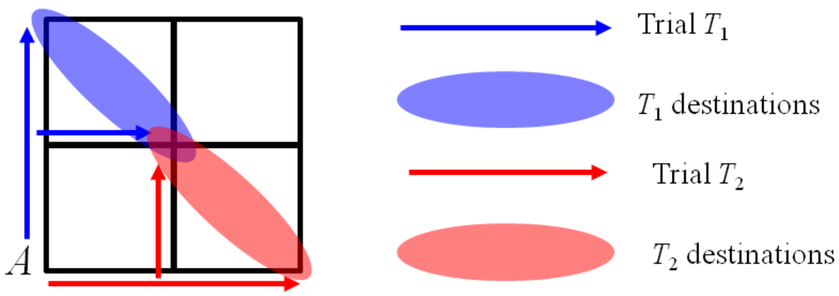
T_1 has $\binom{n-1}{k-1}$ outcomes: the remaining $k-1$ elements of U are chosen from the $n-1$ elements of $X \backslash u$; T_2 has $\binom{n-1}{k}$ outcomes: all k elements of U are chosen from the $n-1$ elements of $X \backslash u$.

The outcomes of T_1 and T_2 are exclusive since either $u \in U$ or $u \notin U$. So

$$\#\text{choices of } U = \#\text{outcomes of } T_1 + \#\text{outcomes of } T_2 = \binom{n-1}{k-1} + \binom{n-1}{k}.$$

Note the convenient shorthand '#' for 'number of'.

Why do T_1 and T_2 have to have exclusive outcomes? Suppose we were asked how many **destinations** can be reached by a 2-step walk in our earler 3×3 grid. We might say T_1 = destinations reached after an initial v step; T_2 = destinations reached after an initial h step. T_1 and T_2 both have two destination outcomes but these are **not** exclusive and the total number of destinations is **not** $2 + 2$. This is depicted below.



Non-exclusivity is a major source of error in combinatorial counting. It is often referred to as 'double counting'. In the above example we double count one of the destinations if we give the answer 4.

- Double counting is also used as a term to describe a clever way of proving things — we will meet this different meaning in the next lecture.
- We can control for double counting errors by the use of the **Inclusion-Exclusion Principle** which we also meet in the next lecture.