# Chapter 1. Section 2

## What is Combinatorics?

Combinatorics is the study of **discrete structures** (discrete $\approx \mathbb{Z}$ as opposed to continuous $\approx \mathbb{R}$ or $\mathbb{C}$).

We propose some kind of structure and we ask:

**Existence:** Do such structures actually exist?

**Construction:** If so can we find one?

**Enumeration:** Can we list all of them, or at least say how many there are?

### Examples

1. Structure = $3 \times 3$ matrices with entries in $\{0, 1\}$ and each row and column has at most one 1. Here is one:

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and we note that if *every* row and column has a single 1 then this is the same as a **permutation**:

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \begin{array}{ccc} 1 & \longrightarrow & 2 \\ 2 & \longrightarrow & 1 \\ 3 & \longrightarrow & 3 \end{array}$$

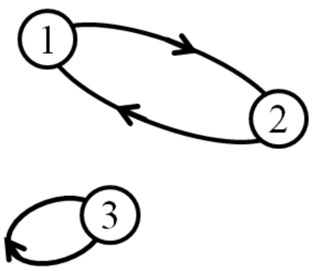
In fact the matrix is called a **permutation matrix**: there is a **bijection** between permutations and these matrices. Permutations have a star role in combinatorics. We can count them: there are

$$n! = n \times n - 1 \times n - 2 \times \ldots \times 2 \times 1$$

permutations of $n$ numbers or objects.

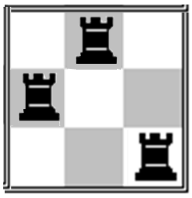
So there are $3! = 3 \times 2 = 6$ matrices over $\{0, 1\}$ in which each row and column has exactly one 1.

2. We could do something else with our matrix: suppose it records connections between three **vertices**:




This is called a **(directed) graph**. What do graphs of permutations look like? How many different 'looking' graphs can we get this way?

3. Our question allowed some rows and columns to be all zeros. This suggests a related classic of combinatorics, **rook placements**: how many ways can we place $k$ rooks on an $n \times n$ chessboard so that no two attack each other (rooks attack horizontally and vertically, i.e. along rows and columns).

   Chapter 2, Section 1).

   We can give the whole answer in one single mathematical object, a **rook polynomial**:

   $$6x^3 + 18x^2 + 9x + 1.$$

   This is a **generating function**: a polynomial in which the coefficients count objects. Here the coefficient of $x^k$ counts placements of $k$ rooks. Thus:

   $x^3$ has coefficient 6 because three rooks is the same as permutations;

   $x^0$ has coefficient 1 because there is one way to put nothing on the chessboard;

   $x^1$ has coefficient 9 because the chessboard has 9 squares and a single rook can go anywhere. For $x^2$ the coefficient of 18 is not so obvious — this is beginning to look like real combinatorics!

   By the way, although combinatorics is discrete, a polynomial also belongs to the continuous world. And in fact rook polynomials like the one above solve differential equations! Actually a variant of the above,

   $$\frac{1}{6}\left(-x^3 + 9x^2 - 18x + 6\right)$$

   solves the **Laguerre equation**

   $$x\frac{d^2y}{dx^2} + (1 - x)\frac{dx}{dy} + 3y = 0,$$

   which appears in quantum mechanics.

4. Back to our matrices but replace $\{0, 1\}$ with $\{1, 2, 3\}$. Now we want *no* number to appears more than once in each row and column:

   | | | |
   |---|---|---|
   | 1 | 2 | 3 |
   | 2 | 3 | 1 |
   | 3 | 1 | 2 |

   This is called a **Latin square** and it is a big deal in experimental design (e.g. we divide up a field into 9 regions and treat the regions with three pesticides as specified by the Latin square; if a busy polluting road runs along the left-hand side of the field we are guaranteed that each pesticide feels its effects, and each pesticide can also be monitored on the unpolluted, right-hand, side of the field).

   What about a $9 \times 9$ Latin square, with entries in $\{1, 2, \ldots, 8, 9\}$? Here is one:

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| 5 | 6 | 4 | 8 | 9 | 7 | 2 | 3 | 1 |
| 9 | 7 | 8 | 3 | 1 | 2 | 6 | 4 | 5 |
| 4 | 5 | 6 | 7 | 8 | 9 | 1 | 2 | 3 |
| 8 | 9 | 7 | 2 | 3 | 1 | 5 | 6 | 4 |
| 3 | 1 | 2 | 6 | 4 | 5 | 9 | 7 | 8 |
| 7 | 8 | 9 | 1 | 2 | 3 | 4 | 5 | 6 |
| 2 | 3 | 1 | 5 | 6 | 4 | 8 | 9 | 7 |
| 6 | 4 | 5 | 9 | 7 | 8 | 3 | 1 | 2 |

It is a pretty special kind of Latin square and belongs (of course) to what is known as 'recreational mathematics' (a field with has a long and distinguished history!) Notice, by the way, that if we replace every number except 1 by zero then we get ... a $9 \times 9$ permutation matrix!

How many Latin squares are there? Nobody knows. How many sudoku puzzles? Estimates are, a bit more than $10^{21}$, nobody knows. Most counting questions in combinatorics are **difficult!**

5. A last example: let's put the set $\{1, 2, \ldots, 8, 9\}$ into our $3 \times 3$ matrix:

| | | |
|---|---|---|
| 8 | 1 | 6 |
| 3 | 5 | 7 |
| 4 | 9 | 2 |

Again every row and every column has a special property: the entries sum to the same value, 15. So do the diagonals: this is called a **magic square** and also belongs to recreational mathematics. Nobody knows how many magic squares there are either, once you get past, say, $10 \times 10$.