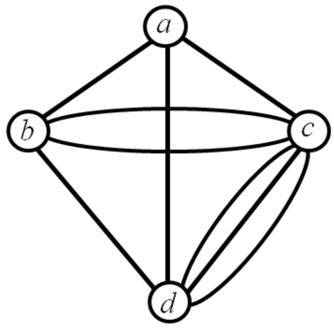
Solutions to Question set for Chapter 7

1. The aim is to count walks and spanning trees for the graph G shown on the right.



   (a) Specify the graph using

   (i) set notation; (ii) its adjacency matrix $A(G)$.

   (b) Calculate

   (i) $A(G)^2$; (ii) $A(G)^3$.

   (c) Explain why the entry a_{11} in the 1st row and column of $A(G)^3$ is 12; and why the entry $a_{1,2}$ is 17. You need not list all the walks involved but you should demonstrate that you know *what* these walks are.

   (d) Write down for G the matrix $L(G)$ that is used by the Matrix Tree Theorem (see Week 8, Lecture 3).

   (e) Calculate the determinant of $L(G)(1\,|\,1)$ and hence determine the number of spanning trees of G.

   (f) Confirm that your answer from part (e) is correct. You need not list all the spanning trees involved but you should demonstrate, e.g. diagrammatically, that you know *what* these spanning trees are. *[HINT: work out separate enumerations for the different ways that you can include edges incident with vertex a in a spanning tree.]*

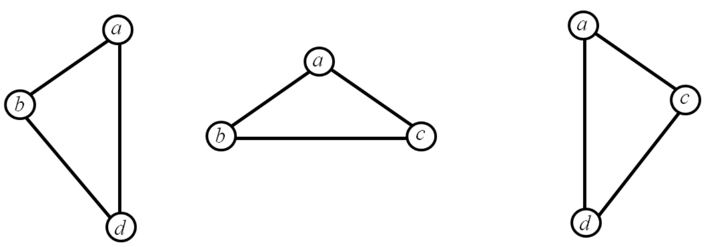
SOLUTION:

(a) (i) $G = (V, E)$, where $V = \{a, b, c, d\}$ and $E = \{ab, ac, ad, bc, bc, bd, cd, cd, cd\}$.

(ii) $A(G) = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 2 & 1 \\ 1 & 2 & 0 & 3 \\ 1 & 1 & 3 & 0 \end{pmatrix}$.

(b) $A(G)^2 = \begin{pmatrix} 3 & 3 & 5 & 4 \\ 3 & 6 & 4 & 7 \\ 5 & 4 & 14 & 3 \\ 4 & 7 & 3 & 11 \end{pmatrix}$; $A(G)^3 = \begin{pmatrix} 12 & 17 & 21 & 21 \\ 17 & 18 & 36 & 21 \\ 21 & 36 & 22 & 51 \\ 21 & 21 & 51 & 20 \end{pmatrix}$.

(c) There are 12 walks of length 3 from vertex a to itself, which are the different ways to walk around the following three cycles:



The first can be traversed in **2** ways (clockwise or anticlockwise); the second in **4** ways since edge bc has multiplicity 2; the third in **6** ways since edge cd has multiplicity 3. So total #walks $= 2 + 4 + 6 = 12$.

There are 17 walks of length 3 from vertex a to vertex b. Taking account of edge multiplicities these are:

$$\begin{array}{llll} acdb \times 3, & adcb \times 3 \times 2, & abdb \times 1, & adab \times 1, \\ acab \times 1, & abcb \times 2 \times 2, & abab \times 1. & \end{array}$$

(d) $L(G) = \begin{pmatrix} 3 & -1 & -1 & -1 \\ -1 & 4 & -2 & -1 \\ -1 & -2 & 6 & -3 \\ -1 & -1 & -3 & 5 \end{pmatrix}.$

(e) Using Gaussian elimination, remembering to divide the result by α if a row is multiplied by α:

$$\begin{aligned} \det L(G)(1\,|\,1) &= \det\begin{pmatrix} 4 & -2 & -1 \\ -2 & 6 & -3 \\ -1 & -3 & 5 \end{pmatrix} \\ &= \frac{1}{2} \times \frac{1}{4} \times \det\begin{pmatrix} 4 & -2 & -1 \\ -4 & 12 & -6 \\ -4 & -12 & 20 \end{pmatrix} \\ &= \frac{1}{8} \det\begin{pmatrix} 4 & -2 & -1 \\ 0 & 10 & -7 \\ 0 & -14 & 19 \end{pmatrix} \quad \text{(adding row 1 to rows 2 and 3)} \\ &= \frac{1}{8} \times 4 \times \det\begin{pmatrix} 10 & -7 \\ -14 & 19 \end{pmatrix} = \frac{1}{2}(10 \times 19 - (-7) \times (-14)) = 46. \end{aligned}$$

(Alternatively, you can use the Rule of Sarrus to evaluate a 3×3 (or even a 4×4) determinant. See: Theorem of the Day.)

(f) Here are the different possibilities for using edges from a (which add by the Summation Principle):

No edges incident with a: this is impossible since any spanning tree must include all vertices. Total=0.

One edge incident with a: then we must take two edges from the cycle $bcdb$. No choice of two edges creates a cycle, so for each of the 3 edges incident with a we have:

$$bc, bd : \ 1 \times 2 \text{ choices}; \quad bc, cd : \ 2 \times 3 \text{ choices}; \quad bd, cd : \ 1 \times 3 \text{ choices}.$$

Total: $3 \times (2 + 6 + 3) = 33$ spanning trees.

Two edges incident with a: then we can only take one edges from the cycle $bcdb$, the choices being:

ab, ad : bc or cd =5 choices; ab, ac : bd or cd =4 choices; ac, ad : bc or bd =3 choices.

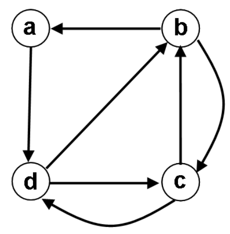
Total: $5 + 4 + 3 = 12$ spanning trees.

Three edges incident with a: then we take no more edges and there is just one spanning tree.

So the final count is: $0 + 33 + 12 + 1 = 46$ spanning trees.

2. A directed graph or **digraph** is a graph with an **orientation** (i.e. arrow) on each edge. Most of the definitions for graphs also apply, in a directed sense, for digraphs: a **walk** is a consecutive sequence of edges in which all the arrows 'go the same way', etc. The degree of a vertex v is split between **indegree**: edges directed into v; and **outdegree**: edges directed out of v.

Suppose $G = (V, E)$ is a digraph. A **directed Euler tour** is a directed walk which visits each edge exactly once and returns to its start vertex. A classic theorem says that G has a directed Euler tour if and only if every vertex has equal indegree and outdegree: $\mathrm{d}^+(v) = \mathrm{d}^+(v) = \mathrm{d}(v)$, say, for all $v \in V$. We then say that G is an *Eulerian digraph*.



For example, in the digraph on the right, the walk $a\ d\ c\ d\ b\ c\ b\ a$ is a directed Euler tour.

A theorem known as the BEST Theorem (see Theorem of the Day) says that the number of distinct directed Euler tours in an Eulerian digraph G is given by the product

$$t_x(G)\prod_{v\in V}(d(v)-1)!$$

where x is any fixed vertex of G and $t_x(G)$ is the number of spanning trees in G in which every vertex has a path directed towards x.

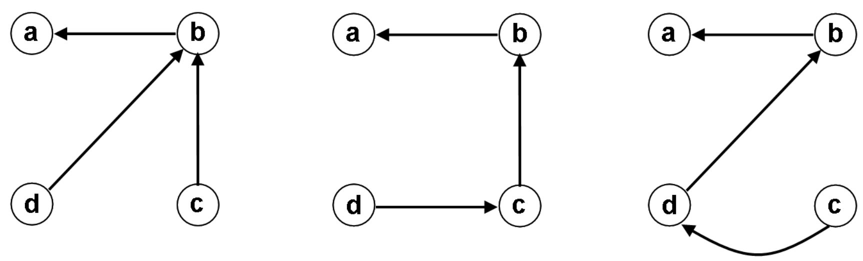
(a) In the digraph above, $t_a(G) = 3$. Find the three spanning trees directed towards a.

(b) For the formula to be well-defined it is necessary that $t_x(G)$ has the same value for any vertex x in an Eulerian digraph G. Confirm that $t_c(G)$ also has value 3, for our directed graph G.

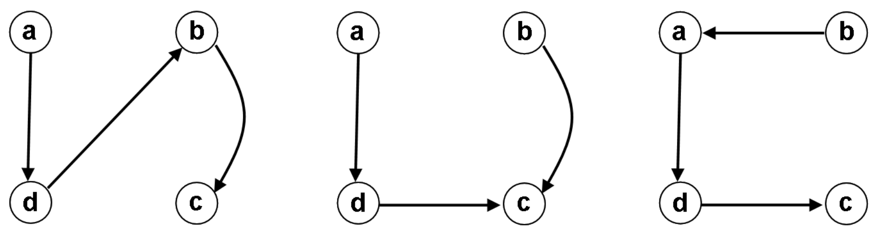
(c) Calculate the number of directed Euler tours in G using the BEST Theorem and find all the tours.

SOLUTION:

(a) The three spanning trees directed towards vertex a are:



(b) The three spanning trees directed towards vertex c, confirming $t_c(G) = t_a(G)$ are:



(c) The product in the BEST Theorem is

$$3 \times (d(a) - 1)! \times (d(b) - 1)! \times (d(c) - 1)! \times (d(d) - 1)! = 3 \times 0! \times 1! \times 1! \times 1! = 3.$$

We have one tour given to us: $a\ d\ c\ d\ b\ c\ b\ a$; the other two are

$$a\ d\ b\ c\ d\ c\ b\ a$$

and $$a\ d\ c\ b\ c\ d\ b\ a$$