Question set for Chapter 6

1. We will use the method of generating functions to solve a degree 3 linear homogeneous recurrence relation (which we could equally solve by the Chapter 3 approach, but now we get something extra: a closed form for the generating function of the recurrence).

   Let $(a_n)_{n\geq 0}$ be defined by

   $$a_n = 8a_{n-1} - 21a_{n-2} + 18a_{n-3},\ n \geq 3, \qquad a_0 = -1,\ a_1 = 1,\ a_2 = 4. \tag{1}$$

   (a) Use equation (1) to write down the generating function $A(x)$ for the sequence $(a_n)_{n\geq 0}$, up to the term in x^5.

   (b) Suppose $C(x)$ is the generating function defined by

   $$c_n = a_n - 8a_{n-1} + 21a_{n-2} - 18a_{n-3} \tag{2}$$

   and that $C(x)$ can be expressed as $A(x)D(x)$ for some unknown generating function $D(x) = \sum_{k=0}^{\infty} d_k x^k$.

   (i) Write down an expression for c_n as a sum of products of the a_ks and d_ks (see Week 6, Lecture 1 notes).

   (ii) By equating terms with (2), find the values of d_k, $k = 0, 1, 2, 3$. Hence, by applying your expression from part (i) for $n = 0, 1, 2$, find the values of c_0, c_1 and c_2.

   (iii) Explain briefly why $c_n = 0$ for $n > 2$, and $d_n = 0$ for $n > 3$.

   (c) Use your answers from part (b) to express $A(x)$ in closed form, as a ratio of two polynomials: a **rational function**.

   (d) Assuming your answer to part (c) is correct, the rational function you found may be expressed, via factorisation and partial fractions separation, as

   $$\frac{-11}{1-2x} - \frac{7}{3(1-3x)^2} + \frac{37}{3(1-3x)}.$$

   (You can check if you have time, by putting the above sum over a common denominator which you then expand out).

   Use the binomial theorem to produce a closed formula for a_n, $n \geq 0$, and check that this gives the same generating function $A(x)$ as you wrote down in part (a).

2. There are two related parts, using exponential generating functions to count quite different combinatorial objects:

   (a) Firstly, we get an exponential generating function (egf) for the number of 2-partions of $[n]$, i.e. for the numbers $S(n, 2)$, where $S(n, k)$ are the Stirling numbers of the 2nd kind (see Week 6, Lecture 3).

   (i) Write down the closed form egf for nonempty subsets of $[n]$.

   (ii) Use your answer to (i) to get the egf for 2-partitions, dividing $[n]$ into *two non-empty* subsets. Make sure that your modification also adjusts for the fact that 2-partitions are not *ordered*, so that $\{1, 3\}, \{2, 4, 5\}$ and $\{2, 4, 5\}, \{1, 3\}$ are the *same* 2-partition of $[5]$.

(iii) By expanding out brackets in your answer to part (ii), and then by writing out the appropriate power series expansions, give a closed formula for $S(n, 2)$.

(b) Now we get the egf for permutations which are **involutions**, i.e. whose squares are the identity permutation. An example on [5] would be 2, 1, 3, 4, 5, since swapping 1 and 2, and then swapping them again, takes us back to the start: 1, 2, 3, 4, 5.

Involutions are easier to recognise in **disjoint cycle notation** in which the above permutation would be written (1 2)(3)(4)(5) (or just (1 2) if we ignore cycles of length 1). For instance, there are four involutions on [3]:

$$(1)(2)(3),\ (1\ 2)(3),\ (1\ 3)(2),\ \text{ and } (1)(2\ 3).$$

(i) Write down the **ten** involutions on [4] in disjoint cycle notation.

(ii) What is the egf for (trivially) arranging a set as a 1-element set or a 2-element set. (Hint: look at equation (3) in the Week 6, Lecture 3 notes).

(iii) Fix N. What is the egf for partitioning $[n]$ into N subsets of sizes 1 or 2. Make sure you adjust for the fact that the partition is *unordered* (see part (a)(ii)).

(iv) Summing over all N, you should be able to write your answer to part (iii) as a product of two exponential series. By expanding out enough terms of this product, confirm that your egf correctly counts involutions on [4].

3. Generating functions give a nice way to prove a truly iconic result, due to Leonhard Euler: $1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \ldots = \frac{\pi^2}{6}$. In other words, if $L(x) = \sum_{k=1}^{\infty} \frac{x^k}{k^2}$, then we want to prove $L(1) = \frac{\pi^2}{6}$.

(a) Use the standard power series expansion $\ln(1-t) = -t - \frac{t^2}{2} - \frac{t^3}{3} - \ldots$ to show that $L(z) = \int_0^z \frac{-\ln(1-t)}{t}\, dt$. (We have switched from x to z because there are complex numbers lurking in the background here...)

(b) The integrand in (a) does not have an antiderivative (using elementary functions). So the integral is labelled a ‘special function’ and given a name: the **dilogarithm**, denoted by $\mathrm{Li}_2(z)$. So now we know that $L(z) = \mathrm{Li}_2(z)$ and, in particular, $L(1) = \mathrm{Li}_2(1)$. Euler had an identity involving these too:

$$\mathrm{Li}_2\left(\frac{-1}{z}\right) + \mathrm{Li}_2(-z) + \mathrm{Li}_2(1) = -\frac{1}{2}(\ln z)^2.$$

Use this to express $\mathrm{Li}_2(1)$ in terms of $\ln(-1)$.

(c) Use another Euler classic, $e^{i\pi} + 1 = 0$, to turn your answer to (b) into the identity $\mathrm{Li}_2(1) = \frac{\pi^2}{6}$.