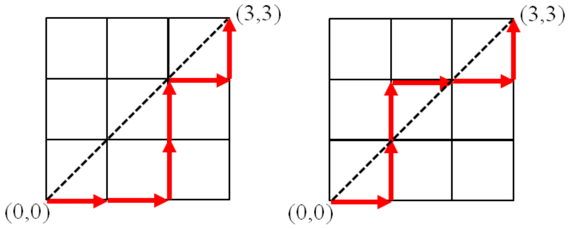
Solutions for Question set for Chapter 4

1. Let L_n denote the square lattice of integer points from $(0,0)$ up to (n,n).

   In L_n, a **Dyck path**[1] is a path from $(0,0)$ to (n,n) consisting of unit steps to the right **R** and upwards **U** such that every point (x,y) on the path satisfies $y \le x$ (i.e. lies on or below the diagonal).
   In the L_3 example on the right, the path on the left is a Dyck path (named after Walther Franz Anton von Dyck (1856–1934), who is also famous for introducing the idea of a free group (to which Dyck paths are closely related.) The path on the right is not a Dyck path, since it climbs to $(1,2)$ which is above the diagonal.

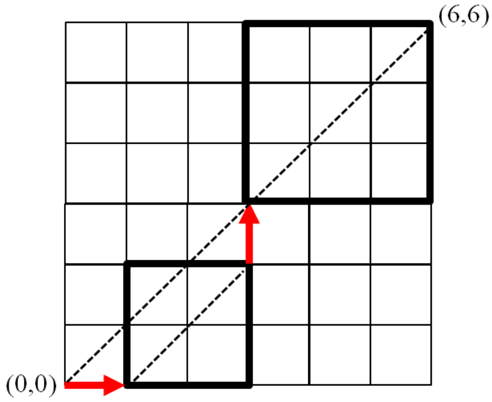
   

   The number of Dyck paths in L_n is equal to the n-th Catalan number C_n. We are going to prove this.

   (a) Draw copies of each of L_1, L_2 and L_3 to show that $C_1 = 1, C_2 = 2$ and $C_3 = 5$ correctly count the number of Dyck paths in these lattices.

   (b) What feature do all your drawings in part (a) share, in terms of **R** and **U** steps (apart from all being Dyck paths, of course)?

   (c) In the drawing of L_6 on the right, the 2×2 and 3×3 heavy squares indicate a further restriction: all Dyck paths from $(0,0$ to $(6,6)$ must be made up of Dyck paths with respect to the heavy squares.

      (i) How many such L_6 Dyck paths do the heavy squares on the right allow?

      (ii) If the first heavy square had size $p\times p$ and the second had size $q\times q$, how many Dyck paths would this allow? You may assume that C_n counts L_n Dyck paths here.

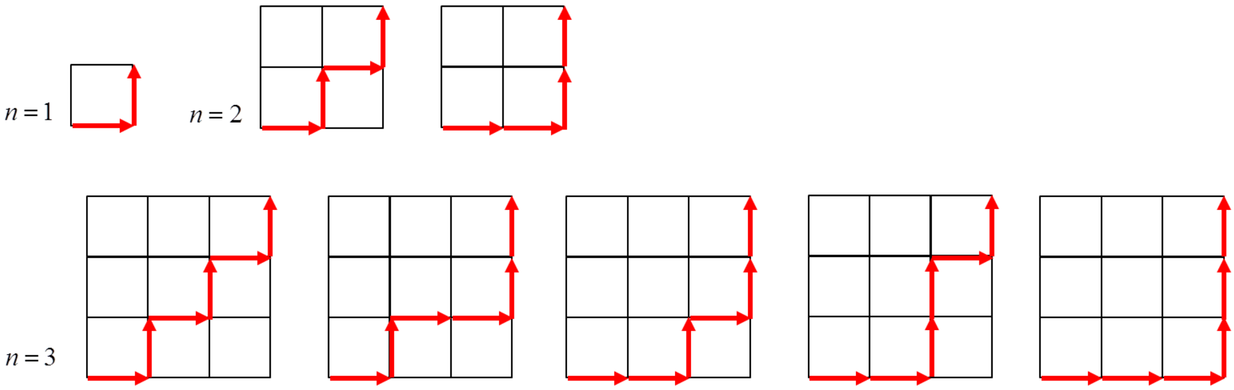
      (iii) What feature do all the Dyck paths in part (i) have, with respect to the diagonal line $(0,0)$ to $(6,6)$ (apart from all being Dyck paths)?

   (d) Give a series of five drawings of L_5, each using two heavy boxes to count Dyck paths which are different with respect to the feature in part (c)(iii). Hence find a relationship between the total number of Dyck paths in L_5 and the values of $C_0, C_1, \ldots, C_4$.

   (e) Sketch an inductive proof that L_n Dyck paths are counted by C_n (you need not be too careful about the wording — just 'give the story'.)

SOLUTION:

(a) The drawings should be something like the following:



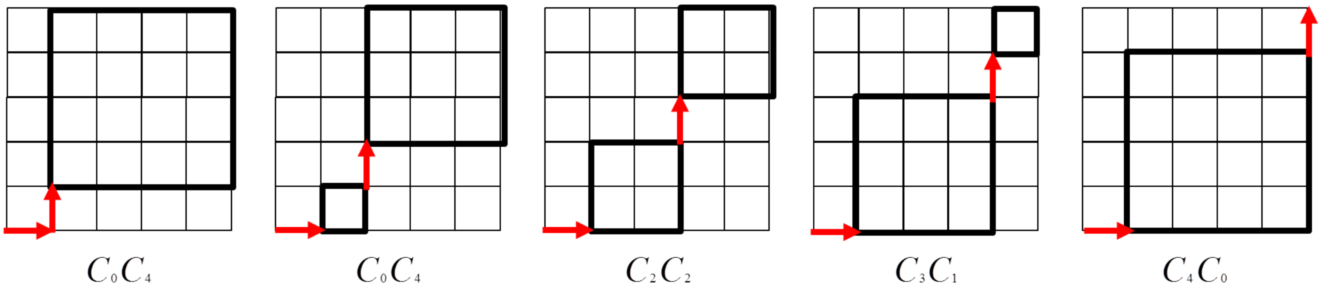
(b) All Dyck paths must begin with a **R** step and end with a **U** step.

(c) (i) There are 2 paths through the 2×2 square and 5 paths through the 3×3 square. So by the Product Principle there are $2 \times 5 = 10$ paths in total.

(ii) The $p \times p$ square has C_p paths and the $q \times q$ square has C_q paths so we have C_pC_q Dyck paths.

(iii) All the paths first touch the diagonal line at the point $(3, 3)$.

(d) The drawings should be something like the following:



(e) There is 1 (trivial Dyck path from $(0, 0)$ to $(0, 0)$. and $C_0 = 1$. So the result is true for $n = 0$.

Now assume that the result is true for all $n \leq N$. Let P be the number of L_{N+1} Dyck paths. From the above discussion we know that

$$P = \#L_k \text{ Dyck paths} \times \#L_{N-k} \text{ Dyck paths added for } k = 0 \text{ up to } k = N.$$

By (strong) induction this is equal to $\sum_{k=0}^{N} C_k C_{N-k}$. But this is equal to C_{N+1}. So the result follows by induction on n.

2. Construct an injective mapping from the following (discussed in class) to L_n Dyck paths (see Question 1).

   (a) Bracket sequences of length $2n$.

   (b) Binary trees on n nodes. [Hint: map trees to bracket sequences first. Think nested backets=left subtree; side-by-side (concatenated) brackets=right subtree.]

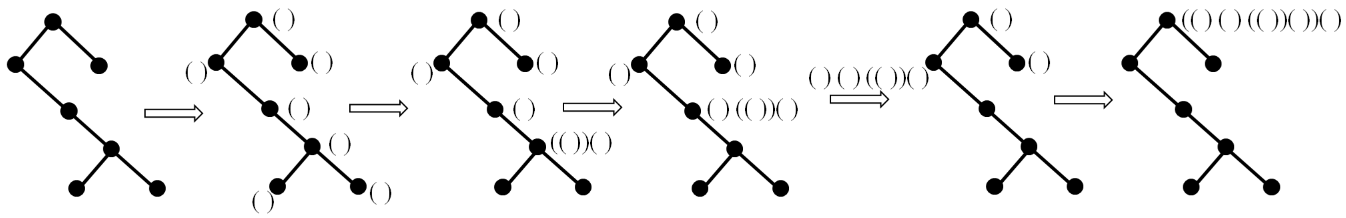
   (c) Vertex-to-vertex triangulations of the regular $(n+2)$-gon. [Hint: map triangulations to binary trees].

   You need not give formal definitions of these mappings. In fact a sufficiently large example (say, $n = 5$) will demonstrate that you know how to do it.

   SOLUTION:

   (a) We note that an L_n Dyck path has $2n$ steps! Just equate '(' with **R** and ')' with **U**. E.g. (())() becomes **RRUURU**.

   (b) Place a bracket pair () on each node of the tree. Now assemble the bracket sequence upwards from the leaves: up from left=nesting; up from right=concatenation. With left tacking precedence over right. Here is an example:



   (c) We fix a 'root' edge of the polygon and identify the unique triangle having this external edge. This is the root vertex of our binary tree. Now we read off triangles to the left and right of this root triangle as giving the left and right subtrees of the root.

   Here are some examples for $n = 4$, with the top edge of the hexagon being the root edge (they are taken from the complete $n = 4$ bijection shown at Theorem of the Day).