Combinatorics

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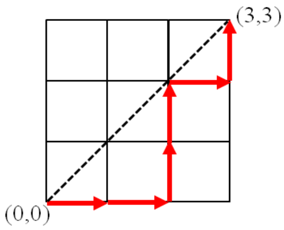
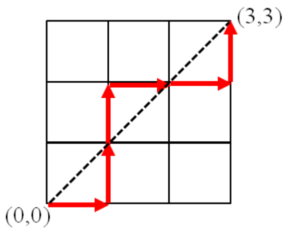
Question set for Chapter 4

1. Let L_n denote the square lattice of integer points from $(0,0)$ up to (n,n).

   In L_n, a **Dyck path**[1] is a path from $(0,0)$ to (n,n) consisting of unit steps to the right **R** and upwards **U** such that every point (x,y) on the path satisfies $y \le x$ (i.e. lies on or below the diagonal).

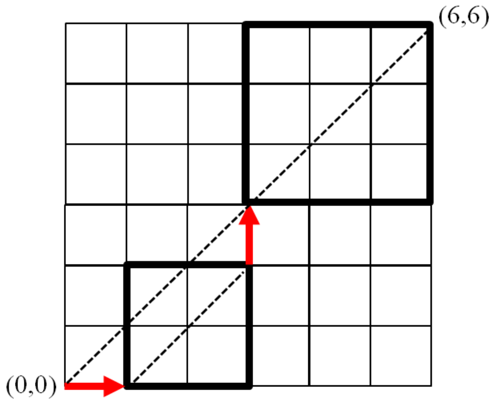
   In the L_3 example on the right, the path on the left is a Dyck path (named after Walther Franz Anton von Dyck (1856–1934), who is also famous for introducing the idea of a free group (to which Dyck paths are closely related.) The path on the right is not a Dyck path, since it climbs to $(1,2)$ which is above the diagonal.

   The number of Dyck paths in L_n is equal to the n-th Catalan number C_n. We are going to prove this.

   (a) Draw copies of each of L_1, L_2 and L_3 to show that $C_1 = 1, C_2 = 2$ and $C_3 = 5$ correctly count the number of Dyck paths in these lattices.

   (b) What feature do all your drawings in part (a) share, in terms of **R** and **U** steps (apart from all being Dyck paths, of course)?

   (c) In the drawing of L_6 on the right, the 2×2 and 3×3 heavy squares indicate a further restriction: all Dyck paths from $(0,0$ to $(6,6)$ must be made up of Dyck paths with respect to the heavy squares.

      (i) How many such L_6 Dyck paths do the heavy squares on the right allow?

      (ii) If the first heavy square had size $p\times p$ and the second had size $q \times q$, how many Dyck paths would this allow? You may assume that C_n counts L_n Dyck paths here.

      (iii) What feature do all the Dyck paths in part (i) have, with respect to the diagonal line $(0,0)$ to $(6,6)$ (apart from all being Dyck paths)?

   (d) Give a series of five drawings of L_5, each using two heavy boxes to count Dyck paths which are different with respect to the feature in part (c)(iii). Hence find a relationship between the total number of Dyck paths in L_5 and the values of $C_0, C_1, \ldots, C_4$.

   (e) Sketch an inductive proof that L_n Dyck paths are counted by C_n (you need not be too careful about the wording — just 'give the story'.)

2. Construct an injective mapping from the following (discussed in class) to L_n Dyck paths (see Question 1).

   (a) Bracket sequences of length $2n$.

   (b) Binary trees on n nodes. [Hint: map trees to bracket sequences first. Think nested backets=left subtree; side-by-side (concatenated) brackets=right subtree.]

   (c) Vertex-to-vertex triangulations of the regular $(n+2)$-gon. [Hint: map triangulations to binary trees.]

   You need not give formal definitions of these mappings. In fact a sufficiently large example (say, $n = 5$) will demonstrate that you know how to do it.