Question set for Chapter 2

1. We will solve the recurrence relation

$$a_n = 3a_{n-2} + 2a_{n-3} - 4n, n \geq 3 \quad (1)$$
$$\text{initial conditions: } a_0 = 1,\ a_1 = 6,\ a_2 = 12. \quad (2)$$

   (a) Calculate the values of a_3, a_4 and a_5.

   (b) Suppose the sequence $b_n = pn + q$ solves the recurrence (1). Find the values of p and q.

   (c) Derive the characteristic equation for the homogeneous equation

$$a_n - 3a_{n-2} - 2a_{n-3} = 0. \quad (3)$$

   (d) Find solutions r_1, r_2 and r_3 for the characteristic equation in part (c) to identify a sequence

$$h_n = \alpha r_1^n + \beta n r_2^n + \gamma r_3^n$$

   which solves the homogeneous equation (3).

   (e) Use the initial conditions (2) to find values of α, β and γ such that $h_n + b_n$ solves (1) and (2). Hence write a final solution $a_n = h_n + b_n$.

   (f) Confirm your solution in part (e) against the values you calculated in part (a).

2. Let D_n, $n \geq 1$, denote the number of derangements of $[n] = \{1, 2, \ldots, n\}$. We will derive and solve a recurrence relation for D_n.

   (a) Let δ be a derangement having the property that δ maps some $k \neq 1$ to 1.

      (i) If there are D_s possible derangements δ which also map 1 back to k, then what is the value of s?

      (ii) Suppose δ does **not** map 1 back to k. If there are D_t possible derangements δ in this case, find the value of t (Hint: suppose 1 is *initially* mapped to k; how many derangements would then rearrange the result to avoid this?)

      (iii) Use parts (i) and (ii) to write D_n as a recurrence relation in terms of D_s and D_t. You will need to factor in the number of possible choices for k.

   (b) Prove that $D_n = n! \sum_{k=0}^{n} \frac{(-1)^k}{k!}$ by showing that this solves the recurrence you found in part (a).