Solutions for Question set for Chapter 2

1. This question studies derangements using the general-n version of Inclusion-Exclusion:

$$\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{k=1}^{n}\left((-1)^{k+1}\sum_{\substack{X\subseteq[n]\\|X|=k}}\left|\bigcap_{j\in X} A_j\right|\right). \tag{1}$$

(a) Write this out in full for $n = 4$: i.e. $|A_1 \cup A_2 \cup A_3 \cup A_4| = \ldots$.

(b) Use your answer to (a) to find an approximate count of the number of primes less than 100 (see the example in Week 2 lecture notes).

(c) Now for derangements: suppose A_i denotes #permutations of $[n]$ which do not move i.

(i) Give an expression in n for $|A_i|$.

(ii) Given an expression in n for $|A_1 \cap A_2|$.

(iii) Given an expression in n and $|X|$ for $\left|\bigcap_{i\in X} A_i\right|$, where $X \subseteq [n]$.

(d) Apply Inclusion-Exclusion (equation 1) to get an expression for the number of permutations of $[n]$ which fix at least one element of $[n]$. (See the count of surjections in Week 2 lecture notes).

(e) Use (d) to show that the proportion of permutations of $[n]$ that are derangements approaches $e^{-1} = 0.367879\ldots$ as $n \to \infty$. (Hint: use the series expansion $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$).

SOLUTION:

(a) The $n = 4$ Inclusion-Exclusion is:

$$\begin{aligned}|A_1 \cup A_2 \cup A_3 \cup A_4| &= |A_1| + |A_2| + |A_3| + |A_4|\\ &\quad -\big(|A_1 \cap A_2| + |A_1 \cap A_3| + |A_1 \cap A_4| + |A_2 \cap A_3| + |A_2 \cap A_4| + |A_3 \cap A_4|\big)\\ &\quad +|A_1 \cap A_2 \cap A_3| + |A_1 \cap A_2 \cap A_4| + |A_1 \cap A_3 \cap A_4| + |A_2 \cap A_3 \cap A_4|\\ &\quad -|A_1 \cap A_2 \cap A_3 \cap A_4|.\end{aligned}$$

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(b) We can test if a number < 100 is composite by checking for divisibility by $2, 3, 5$ and 7. Suppose that A_1 = numbers < 100 divisible by 2; A_2 = numbers < 100 divisible by 3; A_3 = numbers < 100 divisible by 5; and A_4 = numbers < 100 divisible by 7. We get:

$$\begin{aligned}\#< 100 \text{ divisible by } 2,3,5,7 &= |A_1 \cup A_2 \cup A_3 \cup A_4|\\ &\approx \frac{99}{2} + \frac{99}{3} + \frac{99}{5} + \frac{99}{7}\\ &\quad -\left(\frac{99}{2\times3} + \frac{99}{2\times5} + \frac{99}{2\times7} + \frac{99}{3\times5} + \frac{99}{3\times7} + \frac{99}{5\times7}\right)\\ &\quad +\frac{99}{2\times3\times5} + \frac{99}{2\times3\times7} + \frac{99}{2\times5\times7} + \frac{99}{3\times5\times7}\\ &\quad -\frac{99}{2\times3\times5\times7} \approx 115 - 44 + 6 - 0 = 77 \text{ (rounding down).}\end{aligned}$$

So the number of primes $\approx 99 - 77 + 4 = 26$ (the exact number is 25).

(c) (i) $|A_i| = (n-1)!$

(ii) $|A_1 \cap A_2| = (n-2)!$

(iii) $\left|\bigcap_{i\in X} A_i\right| = (n-|X|)!$

(d) Following the lecture notes calculation for surjections, #permutations fixing at least one point is:

$$\begin{aligned}
\left|\bigcup_{i=1}^{n} A_i\right| &= \sum_{k=1}^{n}\left((-1)^{k+1}\sum_{\substack{X\subseteq[n]\\|X|=k}}\left|\bigcap_{j\in X}A_j\right|\right)\\
&= \sum_{k=1}^{n}\left((-1)^{k+1}\sum_{\substack{X\subseteq[n]\\|X|=k}}(n-|X|)!\right)\\
&= \sum_{k=1}^{n}(-1)^{k+1}\binom{n}{k}(n-k)! = \sum_{k=1}^{n}(-1)^{k+1}\frac{n!}{k!}.
\end{aligned}$$

(e) Now the proportion of permutations that fixing one or more points is got by dividing by #permutations $= n!$ to get $\sum_{k=1}^{n}\frac{(-1)^{k+1}}{k!}$. And proportion that are derangements (no point fixed) is:

$$1-\sum_{k=1}^{n}\frac{(-1)^{k+1}}{k!} = \frac{(-1)^0}{0!}+\sum_{k=1}^{n}\frac{(-1)^k}{k!} = \sum_{k=0}^{n}\frac{(-1)^k}{k!} = e^{-1}, \qquad \text{in the limit.}$$

2. We will use the Pigeon Hole Principle to show that a certain function is not $1-1$ and this will prove:

Lemma (Sved–Vázsonyi)[1] Suppose we have a sequence of integers $a_1, a_2, \ldots, a_n$, (not necessarily distinct). Then there is some consecutive subsequence $a_k, a_{k+1}, \ldots, a_{k+s}$ with the sum of its entries being divisible by n.

(a) Check this is true for the sequence $1,\ 3,\ 2,\ 3,\ 3,\ -1,\ 4$ (with $n = 7$).

(b) For the sequence $a_1, a_2, \ldots, a_n$, define the function $f : [n] \to \{0, 1, \ldots, n-1\}$ by $f(m) = a_1 + \ldots + a_m \bmod n$.

(i) Show that f is not $1-1$ on the sequence in part (a).

(ii) Looking back to part (a), what do you notice about the values of m which are mapped by f to the same remainders?

(iii) Prove that, if f does not map any value to zero then it cannot be $1-1$ (Pigeon Hole Principle!)

(c) You should now be able to write out a complete proof of the Lemma (tricky but good practice!)

[1] The great Paul Erdős apparently attributed this lemma to Andrew Vázsonyi and Marta Sved. Sved wrote an interesting 'novel' (*Journey into Geometries*, MAA, 1997) in which she explains non-Euclidean geometry in the form of a follow-up to Lewis Carroll's *Alice in Wonderland*; you can read about Vázsonyi here en.wikipedia.org/wiki/Andrew_Vázsonyi.

SOLUTION:

(a) There are two subsequences: $3+2+3+3-1+4 = 14$ and $2+3+3-1 = 7$.

(b) (i) We get function values:

$$\begin{array}{lll} f(1) = 1 \mod 7 = 1 & f(2) = 1+3 \mod 7 = 4 & f(3) = 4+2 \mod 7 = 6 \\ f(4) = 6+3 \mod 7 = 2 & f(5) = 2+3 \mod 7 = 5 & f(6) = 5-1 \mod 7 = 4 \\ f(7) = 4+4 \mod 7 = 1 & & \end{array}$$

So $f(1) = f(7)$ and $f(2) = f(6)$.

(ii) The two subsequences in part (a) run from $i = 2$ to $i = 7$ and from $i = 3$ to $i = 6$. So the repeated function values match the start (+1) and end of subsequences.

(iii) If f maps no value to 0 then it maps n values to $n-1$ non-zero remainders, so by the Pigeon Hole Principle it must map two values to the same remainder.

(c) **Proof:** If $f(m) = 0$ for some m then $a_1 + \ldots + a_m$ is the required subsequence. So suppose f maps no value to zero. Then by the Pigeon Hole Principle, f maps two values, say m_1 and m_2 to the same remainder, mod n. Assume that $m_1 < m_2$.

Then

$$a_1 + \ldots + a_{m_1} \mod n = a_1 + \ldots + a_{m_2} \mod n$$

but this means that $0 = a_1 + \ldots + a_{m_2} - (a_1 + \ldots a_{m_1}) \mod n$.

Since $m_1 < m_2$ this leaves $0 = a_{m_1+1} + \ldots + a_{m_2} \mod n$ which means that $a_{m_1+1} + \ldots + a_{m_2}$ is divisible by n.