Solutions to Question set for Chapter 10

1. A set system on ground set $E = \{a, b, c\}$ is given as $\mathscr{A} = \{A_1, A_2, A_3\}$ where

$$A_1 = \{b\},\ A_2 = \{a, c\},\ A_3 = \{b\}.$$

   (a) Use Hall's Theorem to show that no complete transversal for $\mathscr{A}$ exists.

   (b) Suppose that A_1 and A_2 are replaced by $\{a, b, c\}$, while A_3 is replaced by $\{b, c\}$. Explain why $\{a, b, c\}$ is a complete transversal of the resulting set system.

   (c) Use the new version of $\mathscr{A}$ from part (b).

      (i) Draw a bipartite graph, with bipartition $X = E$ and $Y = \mathscr{A}$, representing the incidence relation between E and $\mathscr{A}$.

      (ii) Specify a perfect matching for this graph.

      (iii) Write down the 3×3 matrix representing the incidence relation between E and $\mathscr{A}$ and calculate the permanent of this matrix, explaining how this calculation enumerates the perfect matchings of your bipartite graph.

   (d) Give a planar drawing of your bipartite graph from part (c)(i), labelling its vertices $1, \ldots, 6$ in such a way that vertex 1 is adjacent to two vertices: 2 and 6 while vertex 4 is adjacent to two vertices: 3 and 5. Add to your drawing a spanning tree with oriented edges directed away from vertex 1.

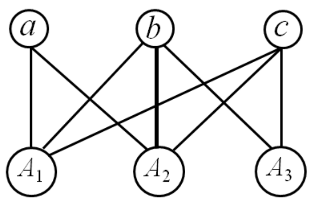
   (e) Apply Kastelyn's Theorem using your drawing in part (d) to enumerate perfect matchings in the graph.

SOLUTION:

   (a) If $X = 1, 3$ then $|X| = 2$ and $|\cup_{i\in X} A_i| = |\{b\} \cup \{b\}| = 1$. So X fails to satisfy Hall's condition and no complete transversal can exist.

   (b) For the new $\mathscr{A}$, $\{a, b, c\}$ is a complete transversal because a can choose A_1, b can choose A_2 and c can choose A_3.

   (c) Using the new version of $\mathscr{A}$:

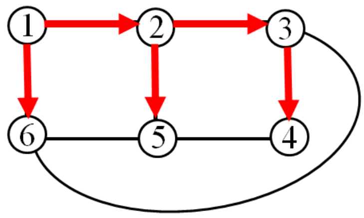


      (i) The bipartite graph is as shown on the right.

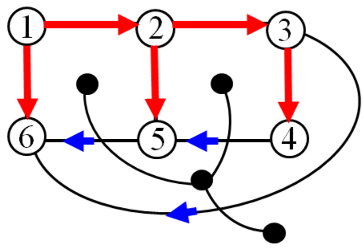
      (ii) A perfect matching is: aA_1, bA_2, cA_3 (there are three others).

      (iii) The matrix is: $M(\mathscr{A}) = \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \end{array}\!\!\begin{array}{c} \begin{array}{ccc} a & b & c \end{array} \\ \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} \end{array}$. Now $\mathrm{per}(M(\mathscr{A})) = 4$ which is the number of subsets of entries which form distinct permutation matrices. Each subset represents a perfect matching.

(d) The graph with one possible spanning tree is as shown on the right.



(e) Add the spanning tree of the dual graph whose edges cross non-spanning-tree edges of the bipartite graph, as shown on the right.
Orient non-spanning-tree edges to give each face an odd number of clockwise edges, as shown.



Now the signed adjacency matrix is shown below left (zeros omitted for clarity). Calculate the determinant: add row 1 to rows 3 and 5, and subtract row 2 from row 6 (below right).

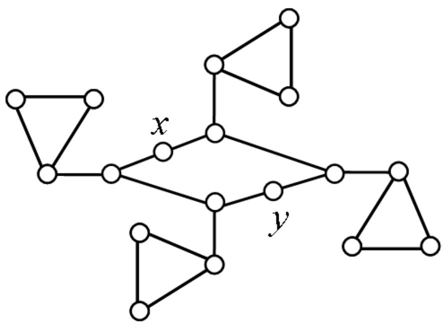
$$
\begin{array}{c} \\ 1\\2\\3\\4\\5\\6 \end{array}
\begin{array}{c}
\begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array}\\
\begin{pmatrix}
 & 1 & & & & 1\\
-1 & & 1 & & 1 & \\
 & -1 & & 1 & & 1\\
 & & -1 & & 1 & \\
 & -1 & & -1 & & 1\\
-1 & & -1 & & -1 &
\end{pmatrix}
\end{array}
\qquad
\begin{array}{c} \\ 1\\2\\3\\4\\5\\6 \end{array}
\begin{array}{c}
\begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array}\\
\begin{pmatrix}
 & 1 & & & & 1\\
-1 & & 1 & & 1 & \\
 & & & 1 & & 2\\
 & & -1 & & 1 & \\
 & & & -1 & & 2\\
 & & -2 & & -2 &
\end{pmatrix}
\end{array}
$$

Now

$$
\begin{aligned}
\det &= -1 \times -1 \times \det\begin{pmatrix} 0 & 1 & 0 & 2\\ -1 & 0 & 1 & 0\\ 0 & -1 & 0 & 2\\ -2 & 0 & -2 & 0 \end{pmatrix} = \det\begin{pmatrix} 0 & 1 & 0 & 2\\ -1 & 0 & 1 & 0\\ 0 & 0 & 0 & 4\\ -2 & 0 & -2 & 0 \end{pmatrix} = -4 \times \det\begin{pmatrix} 0 & 1 & 0\\ -1 & 0 & 1\\ -2 & 0 & -2 \end{pmatrix}\\
&= -4 \times -1 \times \det\begin{pmatrix} -1 & 1\\ -2 & -2 \end{pmatrix} = 16.
\end{aligned}
$$

So by Kastelyn's Theorem the number of perfect matchings is $\sqrt{16} = 4$, consistent with the value found in (c)(iii).

2. Suppose G is a undirected graph and U is a subset of the vertices of G. By the *deletion* of U from G we mean removing from G all the vertices in G together with any edges which are joined to vertices in U. Denote by $\mathrm{odd}(U)$ the number of connected components which are left which have an odd number of vertices, when you delete U from G. One of the most celebrated theorems in graph theory[1] says that G has a perfect matching if and only if, for every subset U of vertices, $\mathrm{odd}(U) \leq |U|$.



(a) Use the theorem to prove that the graph above-right has no perfect matching

(b) Give a combinatorial proof that the graph has no perfect matching.

(c) What if an edge is added joining x to y?

[1] Due to Bill Tutte, 1954

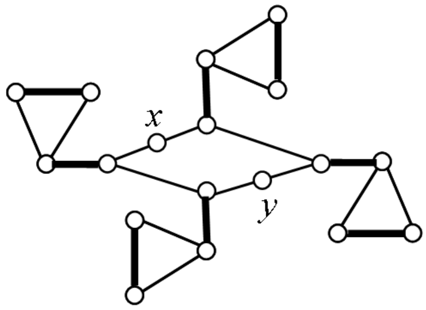
SOLUTION:

(a) Denote by a and b the vertices adjacent to x and by c and d the vertices adjacent to y. Then $U = \{a, b, c, d\}$ is a set of size 4 but if it is deleted from G we are left with 4 triangles (K_3's) and two isolated vertices, x and y. These are a total of six connected components each having an odd number of vertices. So

$$\mathrm{odd}(U) = 6 > 4 = |U|,$$

and Tutte's theorem tells us that there can be no perfect matching.

(b) We try to construct a matching. The heavily-drawn edges in the picture below are forced: they are the only way in which we can span the vertices of the triangles:



Now there is no edge we can add which will span the remaining vertices x and y. So no perfect matching is possible.

(c) If an edge joining x to y is added then this is just what we need to complete the matching in the picture. So now a matching is possible. And indeed, the deletion of our set $U = \{a, b, c, d\}$ no longer leaves six odd components because x and y now form a single even component. So odd(U) becomes 4 which obeys the bound of Tutte's theorem (this does not itself guarantee a matching: we can look for some other set U which disobeys the bound; in fact we know we will not find one, since a perfect matching exists).