Solutions for Question set for Chapter 1

1. A generalised version of the Pigeon Hole Principle is the following:

   Suppose $q_1, q_2, \ldots, q_n$ are positive integers. If $q_1 + q_2 + \ldots + q_n - n + 1$ objects are put into n boxes then for some i, $1 \leq i \leq n$, box i contains at least q_i objects.

   (a) What values of the q_i produce the original version of the Pigeon Hole Principle?

   (b) Use the generalised Pigeon Hole Principle to answer the following question: A basket of fruit is being arranged out of apples, bananas, and oranges. What is the smallest number of pieces of fruit that should be put in the basket in order to guarantee that either there are at least 8 apples or at least 6 bananas or at least 9 oranges?

   (c) Give a proof by contradiction of the generalised Pigeon Hole Principle.

   SOLUTION:

   (a) With $q_1 = q_2 = \ldots = q_n = 2$ the generalised PHP becomes "If $2n - n + 1$ objects are put into n boxes then some box contains more than one object".

   (b) Consider the basket divided into three parts, A, B and O for the different fruits. We want $|A| \geq 8 \vee |B| \geq 6 \vee |O| \geq 9$.

   With $q_1 = 8, q_2 = 6$ and $q_3 = 9$ the generalised PHP guarantees this provided there are $8 + 6 + 9 - 3 + 1 = 21$ pieces of fruit.

   (c) Suppose the generalised PHP is false: we put $q_1 + q_2 + \ldots + q_n - n + 1$ objects are put into n boxes for each i, box i contains fewer than q_i objects.

   Then what is the most we can have if we add up all the boxes? It is

   $$(q_1 - 1) + (q_2 - 1) + \ldots + (q_n - 1).$$

   This is $q_1 + q_2 + \ldots + q_n - n$. But this counts all the objects and we assumed there were $q_1 + q_2 + \ldots + q_n - n + 1$ of them.

   So this is a contradiction and therefore the generalised PHP is true.

2. The *divisor function* $d(n)$, for positive integers n, counts the number of divisors of n. So $d(6) = 4$ because 6 is divided by $1, 2, 3$ and 6. The function behaves rather erratically and is hard to predict (after all n is a prime number if and only if $d(n) = 2$). But the *mean divisor function* $\bar{d}(n) = \dfrac{1}{n}\sum_{k=1}^{n} d(k)$ behaves in a very regular manner as this question shows by double counting.

   (a) Fill in the following table by placing a 1 in row k, column n if and only if k divides n:

	1	2	3	4	5	6	7	8	9
1									
2									
3									
4									
5									
6									
7									
8									
9									

   (b) What does column n count?

   (c) If we count all the 1's in all the columns and divide by 9 what function value do we get?

   (d) How many 1's are there in row k? Express your answer in terms of the floor function $\lfloor x \rfloor$ = largest integer not exceeding x.

   (e) If we added all rows with the table having n rows and n columns what would the total be, in $\sum$ notation?

   (f) Use double counting, and the fact that $\lfloor x \rfloor$ is always less than 1 from the value of x, to give an approximate value for $\bar{d}(n)$.

SOLUTION:

   (a) The completed table is

	1	2	3	4	5	6	7	8	9
1	1	1	1	1	1	1	1	1	1
2		1		1		1		1	
3			1			1			1
4				1				1	
5					1				
6						1			
7							1		
8								1	
9									1

   (b) Column n counts $d(n)$.

   (c) Over the whole table we get $\frac{1}{9}\left(d(1) + \ldots + d(9)\right) = \bar{d}(9)$.

   (d) Row k contains $\lfloor 9/k \rfloor$ 1's.

(e) Total count is $\sum_{k=1}^{n} \left\lfloor \frac{n}{k} \right\rfloor$.

(f) So

$$\begin{aligned} \bar{d}(n) &= \frac{1}{n} \sum_{k=1}^{n} \left\lfloor \frac{n}{k} \right\rfloor \\ &\leq \frac{1}{n} \sum_{k=1}^{n} \frac{n}{k} \\ &= \sum_{k=1}^{n} \frac{1}{k}. \end{aligned}$$

Since the inequality involves an error of less than $\frac{1}{n}$ for each term in the sum, the total error is less than $+1$. So the mean divisor function $\bar{d}(n)$ is always within ± 1 of the sum of the first n reciprocals.

[This sum $\sum_{k=1}^{n} \frac{1}{k}$ is called the n-th narmonic number, denoted H_n, so our estimate can be written $H_n - 1 < \bar{d}(n) < H_n$. H_n is closely approximated by $\log n$ (log to base e) and in fact we get $-1 + \log n < \bar{d}(n) < 1 + \log n$.]