Examples of Recommendations for Improvements to Mathematical Notation

1. In the Preface to his *Elementary Treatise on Determinants*, 1867, reprinted by Rough Draft Printing, 2007, Charles L. Dodgson, refers to "the notation introduced by Leibnitz [sic], $\left\{\begin{array}{lll} a_{1,1}, & a_{1,2}, & \dots \\ a_{2,1}, & a_{2,2}, & \dots \end{array}\right\}$, where the changes, both of column and row, are alike denoted by subscripts. But," he continues,

   > it seems a fatal objection to this system that most of the space is occupied by a number of a's, which are wholly superfluous, while the only important part of the notation is reduced to minute subscripts, alike difficult to the writer and the reader. It was almost an obvious improvement on this system to raise the subscripts into the line, and omit the a's altogether, as suggested by Baltzer, thus — $\left\{\begin{array}{lll} (1,1), & (1,2), & \dots \\ (2,1), & (2,2), & \dots \end{array}\right\}$, and this system, though tedious for writing, might serve very well, were it not for its liability to be confused with the notation, common in Plane Algebraic Geometry, by which $(1,1)$ denotes the Point $x = 1$, $y = 1$. The symbol $1 \rceil 1$, which I have ventured to suggest as an emendation on this last, will be found, I have great hopes, sufficiently simple, distinct, and easy to be written. I have turned the symbol towards the left, in order to avoid all chance of confusion with $\int$, the symbol for integration.

2. Donald Knuth in "Two notes on notation", *American Math. Month.*, 99(5), 1992, 403–422 (online at arxiv.org/abs/math/9205211) recommends replacing the subscripts and superscripts in Σ notation with an inline version borrowed from Kenneth Iverson: for any property $P(k)$ of integers, $\sum_{P(k)} f(k)$ is replaced by $\sum_k f(k)[P(k)]$. The $[\dots]$ notation denotes the characteristic function for P. Here is a nice example, taken from Knuth's paper:

$$\begin{aligned}
\sum_{k\geq 1} \binom{n}{\lfloor \log_2 k \rfloor} &= \sum_{k\geq 1}\sum_m \binom{n}{m} [m = \lfloor \log_2 k \rfloor] \\
&= \sum_{k,m} \binom{n}{m} [m \leq \log_2 k \leq m+1] [k \geq 1] \\
&= \sum_{k,m} \binom{n}{m} \left[2^m \leq k \leq 2^{m+1}\right] [k \geq 1] \\
&= \sum_m \binom{n}{m} \left(2^{m+1} - 2^m\right) [m \geq 0] = \sum_m \binom{n}{m} 2^m = 3^n.
\end{aligned}$$

3. In "Two notes on notation" Knuth cites Charles Babbage's *Passages on the Life of a Philosopher* in chapter 4 of which Babbage describes his efforts to promote Leibniz's 'd' notation: "I then drew up the sketch of a society to be instituted for translating the small work of Lacroix on the Differential and Integral [Calculus]. It proposed that we should have periodical meetings for the propagation of d's ; and consigned to perdition all who supported the heresy of dots." This delightful book is free as an e-book:
books.google.co.uk/books/about/Passages_from_the_life_of_a_philosopher.html
(the quote is in Chapter 4 on page 28).

4. Knuth again: in "Big Omicron and Big Omega and Big Theta", *SIGACT News*, 18, Apr.-June 1976, Knuth proposes that $\Omega(f(x))$ should mean the class of functions which exceed some suitable constant multiple of f for all x large enough. This is indeed now the common usage in,

for example, computer science and extremal combinatorics. But the notation was introduced by Hardy and Littlewood to mean "for infinitely many x, instead of "for *all* large x":

> Although I have changed Hardy and Littlewood's definition of Ω, I feel justified in doing so because their definition is by no means in wide use, and because there are other ways to say what they want to say in the comparatively rare cases when their definition applies. I like the mnemonic appearance of Ω by analogy with O, and it is easy to typeset. Furthermore, these two notations as defined above are nicely complemented by the Θ-notation which was suggested to me independently by Bob Tarjan and by Mike Paterson.

(Notice the typesetting consideration.) The Wikipedia entry on O drily observes "However, the HardyLittlewood definition had been well used for at least 25 years." And of course it is better suited to the kind of $\liminf$ arguments which appear in analytic number theory.

By the way, another notational issue is raised in "Big Omicron...":

> The phenomenon of one-way equalities arises in this connection, i.e., we write $l + O(n^{-1}) = 0(1)$ but not $0(1) = l + O(n^{-1})$. The equal sign here really means $\subseteq$ (set inclusion), and this has bothered many people who propose that we not be allowed to use the = sign in this context. My feeling is that we should continue to use one-way equality together with O-notations since it has been common practice of thousands of mathematicians for so many years now, and since we understand the meaning of our existing notation sufficiently well.

5. Stephen Wolfram has a wide-ranging transcript on: "Mathematical Notation: Past and Future" online www.stephenwolfram.com/publications/recent/. He cites Florian Cajori's "classic book entitled A History of Mathematical Notation" with which I am not familiar but which anybody seriously engaging in debate on mathematical notation ought presumably to know about. It is reprinted by Dover, ISBN 978-0486677668.

6. Murray Bourne, the well-known teacher and math-blogger who runs www.intmath.com/, has this: www.intmath.com/blog/towards-more-meaningful-math-notation/661 in which he proposes a more consistent use of parentheses: thus $f[a + b]$ means "apply the function f to the sum of a and b, while $f(a + b)$ is an expression which may be expanded $fa + fb$. I was reminded of this when Ray Hill gave his 2013 London Mathematical Society Popular Lecture on "Mathematics in the Courtroom": he said something like "you cannot use a notation such as $P(a\,|\,b)$ with a jury because their only exposure to '$(\dots)$' in mathematics is likely to be in the context of algebraic expressions; even before they get to the subtlety of the | they will be confused."

7. blogs.scientificamerican.com/roots-of-unity/2013/09/12/10-trig-functions-youve-never-heard-of/

8. www.math.uri.edu/ merino/spring06/mth562/ShortHistoryComplexNumbers2006.pdf has a nice quote from Cauchy:

   > We completely repudiate the symbol $\sqrt{-1}$, abandoning it without regret because we do not know what this alleged symbolism signifies nor what meaning to give to it.

9. Albert Eagles convenient notation for $\pi/2$ in *The Elliptic Functions As They Should Be. An account, with applications, of the functions in a new canonical form*, Galloway & Porter Ltd; Cambridge, 1958

10. See pballew.blogspot.fr/2014/01/notes-on-history-of-factorial.html on the history of factorial notations by Pat Ballew.

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