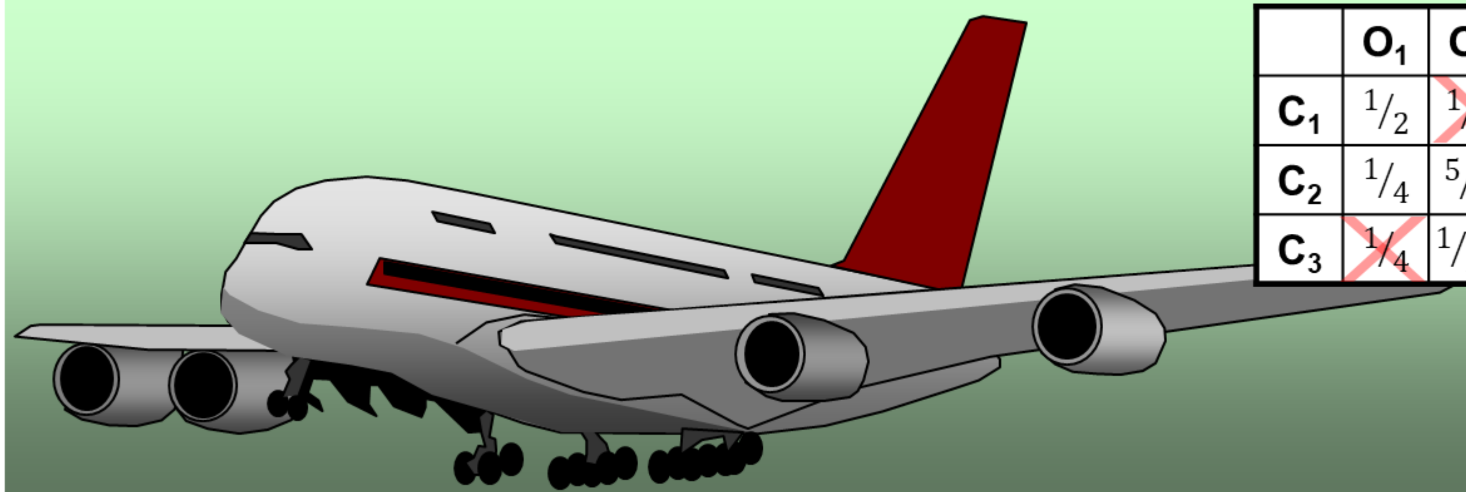
THEOREM OF THE DAY

van der Waerden's Conjecture *If A is an $n \times n$, doubly-stochastic matrix then its permanent satisfies the inequality $\mathrm{per}(A) \geq n!/n^n$, and equality is attained uniquely by the matrix all of whose entries are $1/n$.*



System 1

	O_1	O_2	O_3
C_1	$1/2$	$\cancel{1/3}$	$1/6$
C_2	$1/4$	$5/8$	$\cancel{1/8}$
C_3	$\cancel{1/4}$	$1/24$	$17/24$

System 2

	O_1	O_2	O_3
C_1	$1/2$	$\cancel{1/3}$	$1/6$
C_2	$1/4$	$3/8$	$\cancel{3/8}$
C_3	$1/4$	$\cancel{7/24}$	$11/24$

Modern passenger aircraft are largely computer-controlled. To ensure reliability, outputs to critical functions might typically be given by three computers, with majority voting to eliminate any single erroneous output. In System 1, above top-left, for example, all three computers, C_1, C_2 and C_3, have failed on a single output but majority voting will safely eliminate each error. System 2, however, has potentially suffered a non-recoverable error: if C_1 and C_3 deliver the same incorrect value of O_2 then majority voting will cause this incorrect value to be used. The tables (or matrices) are *doubly stochastic*: each row and each column consists of non-negative entries summing to unity. In this instance, these entries are *relative* probabilities of failure for the three computers on each output. (The actual probabilities of failure would be *much* smaller, operational targets of one failure per billion flying hours being common!)

Which of the two systems is the safer under the condition of three simultaneous computer errors, assuming each computer fails on just one output? The *permanent* of an $n \times n$ matrix is the summation over all products of n elements, one element chosen from each row and column. The failures in System 1 constitute one such product; the permanent of this matrix will enumerate all 'safe' combinations of three failures and will be the probability of safe failure. Its value is $185/576 \approx 0.32$, so relative probability of non-recoverable error is about $1 - 0.32$ or 68%. System 2, on the other hand, has permanent approx. 0.24 and relative failure rate 76%. How bad is this? Van Der Waerden's inequality gives an absolute minimum permanent of $3!/3^3$ or about 0.22, with a relative failure rate of 78%, so that System 2 is fairly close to worst possible.

In the words of Richard A. Brualdi (*Bull. AMS*, 1(6)), "in 1926 van der Waerden ... asked the following question and then quietly walked away ... What is the minimum value of the permanent of an $n \times n$ doubly stochastic matrix?" Serious work was sparked by the proof by Marvin Marcus and Morris Newman, in 1959, that the answer was $n!/n^n$ on the *inside* of the polytope of doubly stochastic matrices. The elimination of rivals on the boundary was published independently by two Russian mathematicians, Georgy Egorychev and Dmitry Falikman, in 1980 and 1981, respectively. In fact, the theorem had essentially been proved by more elementary methods by Béla Gyires in 1977.

Web link: www.jaapspies.nl/mathfiles/dancingschool.pdf.

Further reading: *Nonnegative Matrices and Applications* by R. B. Bapat and T. E. S. Raghavan, Cambridge University Press, 1997.