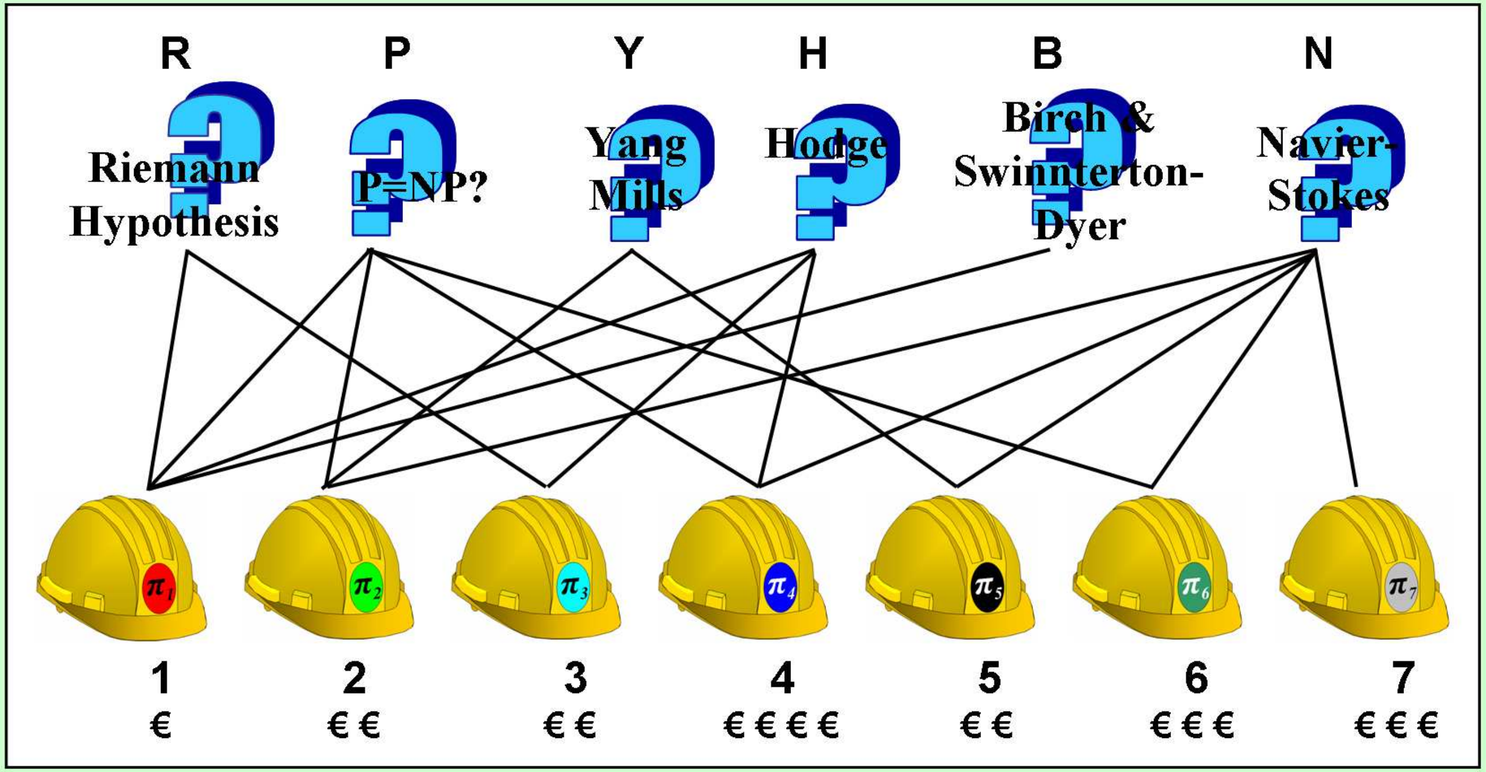
THEOREM OF THE DAY

The Transversal Matroid Theorem *Let E be a finite set and let $\mathcal{A} = \{A_i \mid i = 1, \ldots, t\}$ be a family of subsets of E. Let $\mathcal{T}$ denote the collection of* partial transversals *of $\mathcal{A}$, i.e. those subsets X of E having the property that each element of X may choose a distinct member of $\mathcal{A}$ to which it belongs (more precisely: those subsets $X = \{x_1, \ldots, x_k\}$, $0 \leq k \leq t$, for which there is a subset $\{y_1, \ldots, y_k\}$ of $\{1, \ldots, t\}$ satisfying $x_i \in A_{y_i}$, $i = 1, \ldots, k$). Then $\mathcal{T}$ forms the collection of independent sets of a matroid.*



$$\begin{array}{c} \\ R \\ P \\ Y \\ H \\ B \\ N \end{array} \begin{array}{c} \begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \\ \left(\begin{array}{ccccccc} r_1 & & r_3 & & & & \\ p_1 & p_2 & & p_4 & & p_6 & \\ & y_2 & & & y_5 & & \\ h_1 & & h_3 & h_4 & & & \\ b_1 & & & & & & \\ & n_2 & & n_4 & n_5 & n_6 & n_7 \end{array}\right) \end{array}$$

The above incidence structure shows which workers $(1, 2, \ldots, 7)$ may be assigned to which jobs $(R, \ldots, N)$. The entries, r_1, p_1, etc, are *indeterminates*—they allow us to express any assignment as a 'valueless' product. For instance, the encircled assignment is expressed as $r_3 p_6 y_2 h_4 b_1 n_7$. This assignment gets all 6 jobs done at a cost of €15. But can it be done cheaper?

In the illustration above, each of the six remaining Millennium Problems of the Clay Mathematics Institute is to be assigned to a different subcontractor for Edsger Dijkstra's fictitious Mathematics Inc. company. Obviously we want to solve all six problems as cheaply as possible! The *greedy* approach is to always to take the first, cheapest option: this would give us an assignment starting $R1, P2, Y5, H3$ (or $r_1 p_2 y_5 h_3$, as a product of indeterminates). But now we are stuck because only subcontractor 1 can be assigned to problem B, and we used her for problem R. Thus we may have to backtrack, and this is expressed by saying we cannot match up subcontractors to problems greedily. To say that partial transversals give a matroid is precisely to say that at least a cheapest maximal transversal *can* be selected greedily. That is, we can choose the cheapest subcontractors first: 1 then 2 then 3 then 5, etc, and make the assignment to problems later. This may be more a difference in mathematics than in practice: it leaves open the question of how we know our partial transversals are valid *unless* we find a matching at the same time.

This theorem was proved in the mid-1960s by Jack Edmonds and Delbert Ray Fulkerson in the USA and, independently (and also in an infinite version) by Leon Mirksy and Hazel Perfect at Sheffield University in the UK.

Web link: www.math.lsu.edu/~oxley/dominic.pdf. Read about Dijkstra's Mathematics Inc. at www.cs.utexas.edu/~EWD/ (see e.g. no. 1224).

Further reading: *Introduction to Graph Theory, 5th Ed.*, by Robin J. Wilson, Prentice Hall, 2010, chapter 9.