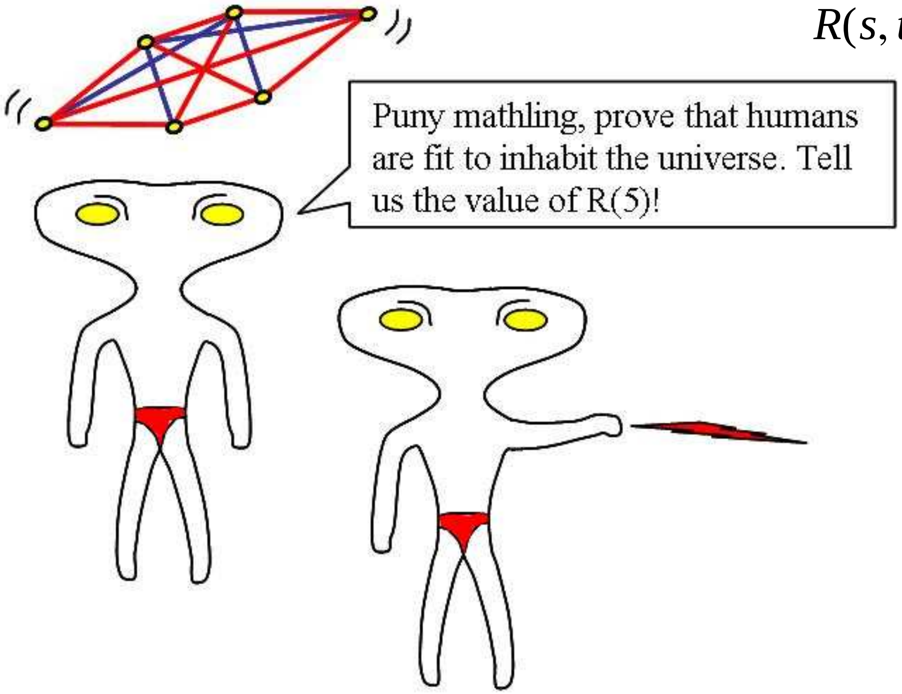
THEOREM OF THE DAY

Ramsey's Theorem *For any pair s and t of positive integers, there is a finite number $R(s,t)$ such that any graph on at least $R(s,t)$ vertices contains an s-vertex independent set or a t-vertex clique. In particular,*

$$R(s,t) \leq \binom{s+t-2}{s-1}.$$



NOTES:

***R*(*s*)**: short for $R(s, s)$

independent set: set of vertices in which no pair is joined

clique: set of vertices in which every pair is joined

K_n: the complete graph, an n-vertex clique

non-trivial being: a mathematician

The theorem says, if $n \geq R(s,t)$ and we colour, red and blue, the edges of a complete graph K_n, then the result must contain either a red K_s or a blue K_t (or both). In the picture, the aliens' spaceship is a complete graph K_6 missing two diagonal edges. Now the value of $R(3) = R(3,3)$ is known to be 6, so if these missing edges are added in either colour, a monochromatic K_3 (i.e. a red or blue triangle) will be created.

Only small values of $R(n)$ are known: for n =1, 2, 3, 4, the values of $R(n)$ are 1, 2, 6 and 18, respectively. The great Paul Erdős reputedly said "If aliens threatened to destroy the Earth unless we gave them the value of $R(5)$ then it would be worthwhile to put all our efforts into finding this number. If they demanded the value of $R(6)$ then we should put all our efforts into killing the aliens." So far only upper and lower bounds for these values can be asserted: $43 \leq R(5) \leq 48$ and $102 \leq R(6) \leq 165$.

Before he died tragically at the age of 26 of liver failure, Frank Plumpton Ramsey made contributions in philosophy, economics and mathematics. His theorem was published in 1930, the year of his death, in a paper entitled *On a Problem of Formal Logic*. It has since given birth to a whole branch of combinatorics called Ramsey Theory. The version of the theorem given here is due to Paul Erdős and George Szekeres.

Web link: math.mit.edu/~fox/MAT307 (see Lecture 5)

Further reading: *Rudiments of Ramsey Theory, 2nd Edition* by Ron Graham and Steve Butler, American Mathematical Society, 2015.