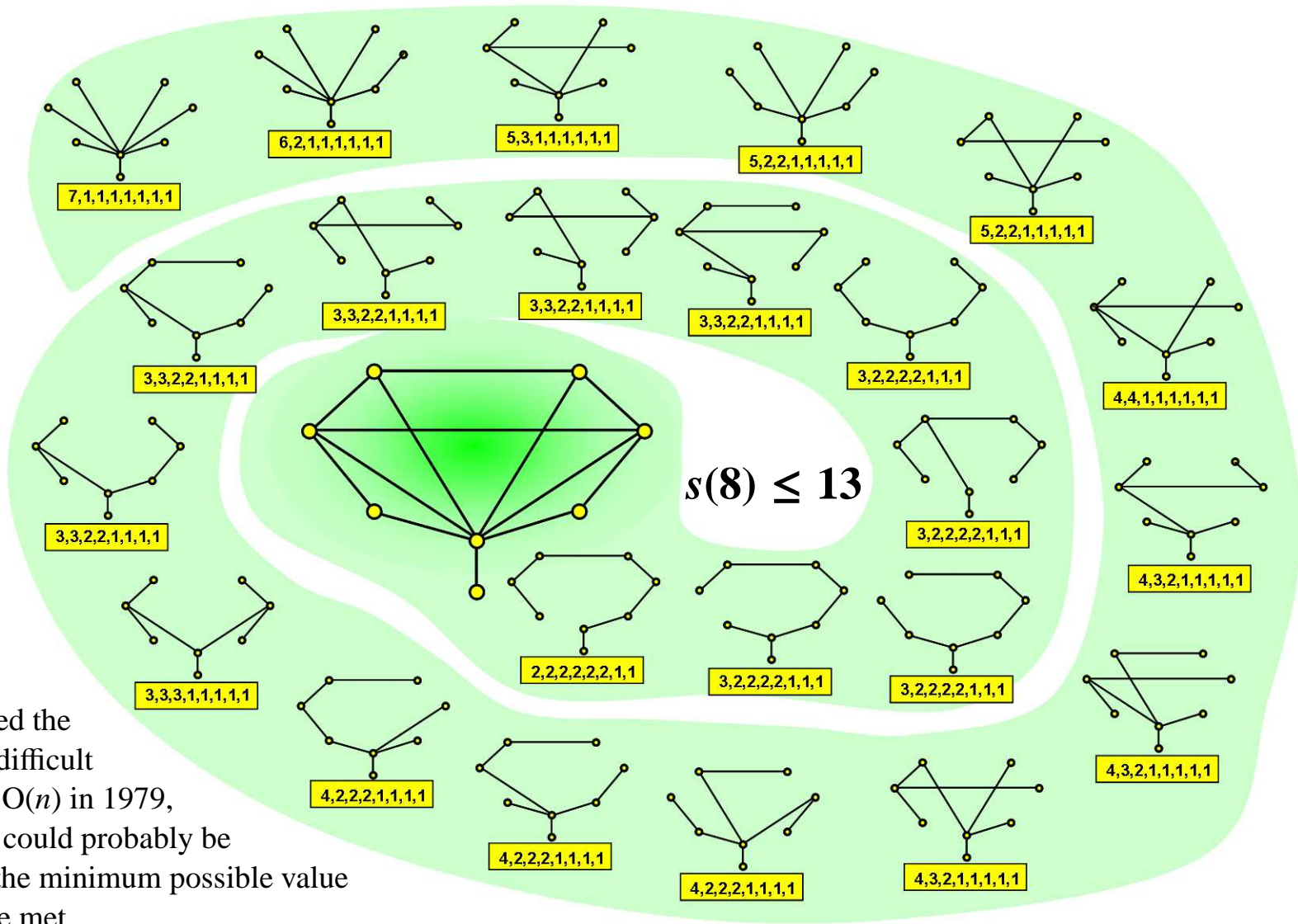




# THEOREM OF THE DAY

**The Panarboreal Formula** Denote by  $s(n)$  the minimum number of edges a graph  $G$  on  $n$  vertices can have so that any tree on  $n$  vertices is isomorphic to some spanning tree of  $G$ . Then  $s(n) \sim cn \log n$  where  $c$  is a constant satisfying  $1/2 \leq c \leq 5/\log 4$ .

There are 23 unlabelled trees having 7 edges; they are shown on the right, lexicographically ordered by degree sequence, together with an 8-vertex graph with 13 edges in which each one may be found as a subgraph. No formula is known for the values of  $s(n)$ , but it is easy to establish that  $s(n) \geq (1/2)(n-1)\log n$ . For, given  $k$ ,  $1 \leq k \leq n$ , we may always choose a tree in whose degree sequence the  $k$ -th entry  $\geq (n-1)/k$  (for example, the 6th tree around the spiral on the right has 2nd entry  $4 \geq (8-1)/2$ ). But now the same must hold for the degree sequence of any graph  $G$  containing this tree. So if  $G$  contains each  $n$ -vertex tree and has degree sequence, say,  $(d_1, \dots, d_n)$ , then the number of edges in  $G$  is  $= \frac{1}{2} \sum_{k=1}^n d_k \geq \frac{1}{2} \sum_{k=1}^n (n-1)/k > \frac{1}{2}(n-1)\log n$ . So  $s(n) > \frac{1}{2}n \log n - O(\log n) \sim \frac{1}{2}n \log n$ .



Fan Chung and Ron Graham proved the easy lower bound on  $s(n)$  and the difficult upper bound of  $(5/\log 4)n \log n + O(n)$  in 1979, mentioning that  $5/\log 4 \approx 3.6067$  could probably be improved, possibly even down to the minimum possible value of  $1/2$ . This challenge has yet to be met.

**Web link:** [math.ucsd.edu/~fan/](http://math.ucsd.edu/~fan/) an Aladdin's cave: all Chung's papers; the one concerned is "On universal graphs for spanning trees", *JLMS*, 1983.  
**Further reading:** *Erdős on Graphs: His Legacy of Unsolved Problems*, by Fan Chung and Ronald Graham, AK Peters, 1998, section 3.5.1.

