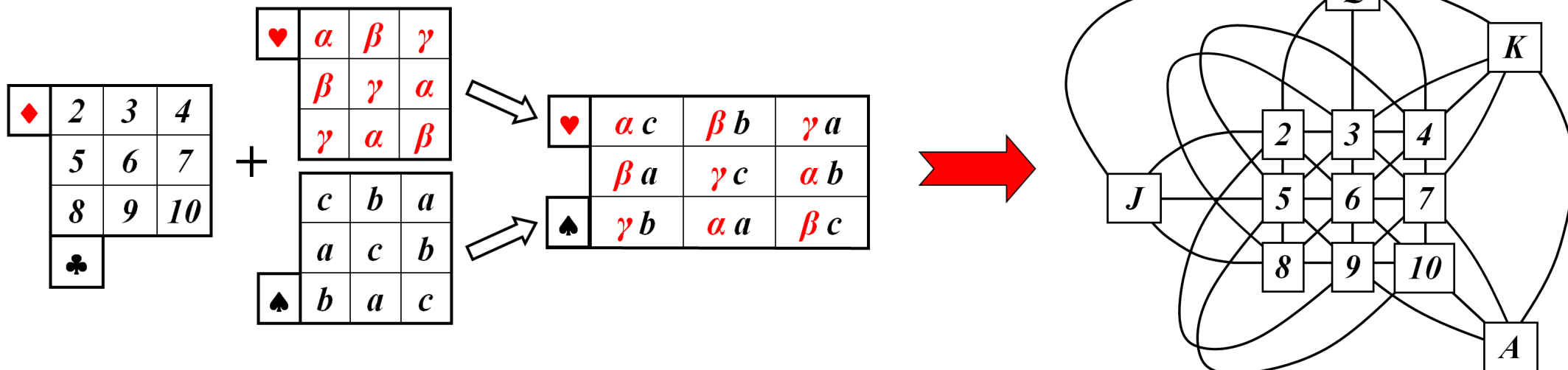




# THEOREM OF THE DAY

**The Bose Equivalence Theorem in Design Theory** *A finite projective plane of order  $n$  exists if and only if a complete set of mutually orthogonal Latin squares of order  $n$  exists.*



**PROBLEM:** Arrange the 52 playing cards of a pack into 13 columns of 4 so that any pair of card values appears in exactly one column, and any pair of columns share exactly one card value.

**SOLUTION:** illustrated in the picture via the ‘if’ part of the theorem, for  $n = 3$ : we arrange the card values 2 through 10 in an  $n \times n$  square (marked here  $\blacklozenge$  and  $\clubsuit$ , although the card suits only relate to our problem not to the theorem). We will combine this with  $n - 1$  mutually orthogonal Latin squares or *MOLS*, marked here  $\heartsuit$  and  $\spadesuit$ . Each *Latin square* has a set of  $n$  symbols appearing exactly once in each row and column; a set of  $n - 1$  forms a *complete set of MOLS* if, superimposed, every cell contains a distinct ordered list of  $n - 1$  symbols (here,  $\alpha c, \beta b, \gamma a$  in the top row, etc).

Each row and column of the  $\blacklozenge\clubsuit$  square becomes a line in the diagram on the right. Additionally, the position values of each Greek letter of the  $\heartsuit$  Latin square are joined with a line (so 2,7,9 for  $\alpha$  etc), and similarly the positions of each Roman letter of the  $\spadesuit$  Latin square. This gives us, so far,  $n^2 + n$  lines, forming a so-called *affine plane* of order  $n$ . In a projective plane, each set of parallel lines meets in a distinct point, added here as  $J, Q, K, A$ . Join these  $n + 1$  points with a line and the projective plane is complete: it has  $n^2 + n + 1$  lines each having  $n + 1$  points;  $n^2 + n + 1$  points each lying in  $n + 1$  lines; every pair of lines meets at exactly one point; every pair of points is joined by exactly one line. We make each line a card column using our chosen suits, and we are done. Click [here](#) to see a version of this page with the solution partially visible in the background (0.6MB).

This beautiful link between geometry and combinatorics is named after Raj Chandra Bose who, in 1938, constructed complete sets of MOLS for all finite fields. The same construction was independently published by Wilfred Leslie Stevens but had been already discovered, it transpires, by Eliakim Moore in 1896.

**Web link:** [www.site.uottawa.ca/~lucia/courses/7160-17/slides/](http://www.site.uottawa.ca/~lucia/courses/7160-17/slides/): click on lecture 5

**Further reading:** *Designs, Graphs, Codes and their Links* by P. J. Cameron and J. H. van Lint, Cambridge University Press, 1991.

