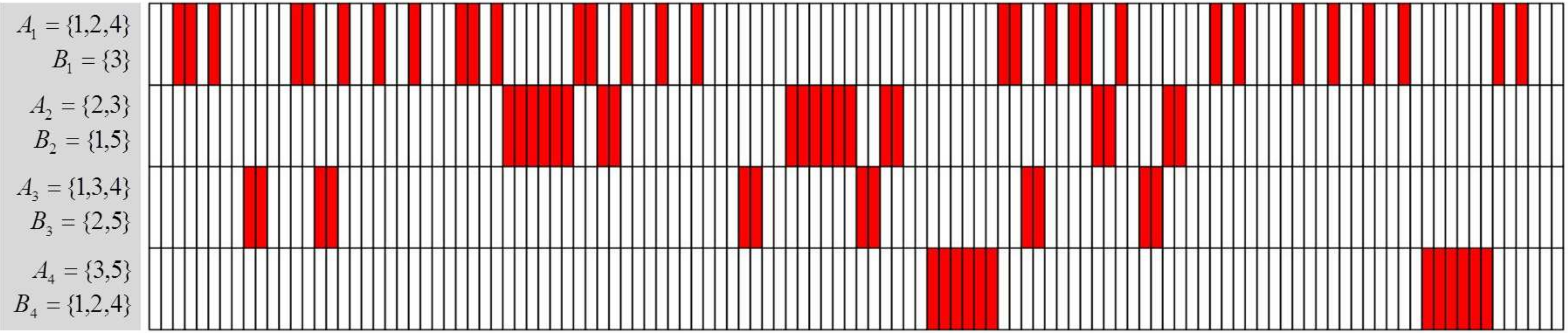
THEOREM OF THE DAY

The LYM Inequality *Let S be a finite set and $\mathscr{F}$ a family of subsets of S none of which is a subset of any other. Then*

$$\sum_{X\in\mathscr{F}} \binom{|S|}{|X|}^{-1} \leq 1.$$

$\mathrm{Sym}(5) = \{[1,2,3,4,5],[1,2,3,5,4],[1,2,4,3,5],[1,2,4,5,3],[1,2,5,3,4],[1,2,5,4,3],[1,3,2,4,5],\ldots,[5,4,1,3,2],[5,4,2,1,3],[5,4,2,3,1],[5,4,3,1,2],[5,4,3,2,1]\}$



Proof. The inequality may be proved from a more general 1965 result of Béla Bollobás:

> *Let S be a finite set of cardinality n and let A_i and B_i, $1 \leq i \leq m$, be sequences of subsets of S having the property that $A_i \cap B_j = \emptyset$ if and only if $i = j$, $1 \leq i, j \leq m$. Then $\sum_{i=1}^{m} \binom{a_i+b_i}{a_i}^{-1} \leq 1$, where $a_i = |A_i|$ and $b_i = |B_i|$, $i = 1, \ldots, m$.*

Indeed, in the LYM inequality, set $m = |\mathscr{F}|$ and list the sets of $\mathscr{F}$ as $A_1, A_2, \ldots, A_m$. Now set $B_i = S \setminus A_i$ for all i. Then $A_i \cap B_j = A_i \cap (S \setminus A_j) = \emptyset$ if and only if $A_i \subseteq A_j$ if and only if $i = j$. To check a small example, consider the rows in the matrix depicted above which are labelled with four pairs of subsets A_i, B_i from the set $\{1,2,3,4,5\}$.

> **Proof of Bollobás's Inequality.** By double counting: form a matrix whose rows are indexed by the pairs A_i, B_i, $i = 1, \ldots, m$, and whose columns are indexed by the $n!$ permutations of $\{1, \ldots, n\}$ (the matrix above illustrates this with $n = 5$ and $m = 4$). Now set the ij-th entry of the matrix to 1 if, in the j-th permutation, all elements of set A_i are listed before all elements of set B_i. Set all other entries to zero. In the matrix above the entries set to 1 are shaded red. For example, the third entry in the first row is red because $A_1 = \{1,2,4\}$, $B_1 = \{3\}$ and the third permutation, $(1,2,4,3,5)$ permutes 3 to come after 1, 2 and 4. There are exactly $\binom{n}{a_i+b_i} a_i! b_i! (n - (a_i + b_i))! = n! \binom{a_i+b_i}{a_i}^{-1}$ permutations which are shaded red in row i. And a permutation may be shaded red in at most one row since if permutation π permutes A_i before B_i and A_j before B_j then $A_i \cap B_j = \emptyset$ or $A_j \cap B_i = \emptyset$, whence $i = j$. So $\sum n! \binom{a_i+b_i}{a_i}^{-1} \leq n!$ and the inquality follows.

With $|S| = n$ in the LYM inequality and using the fact that $\binom{|S|}{|X|} = \binom{n}{|X|} \leq \binom{n}{\lfloor n/2 \rfloor}$ we derive one of the foundational results of combinatorial set theory:

Corollary (Sperner's Theorem, 1928) *Let S be a finite set of cardinality n, and $\mathscr{F}$ a family of subsets of S none of which is a subset of any other. Then $|\mathscr{F}| \leq \binom{n}{\lfloor n/2 \rfloor}$.*

LYM is named for its independent discoverers: David Lubell (1966), Koichi Yamamoto (1954) and Lev Dmitrievich Meshalkin (1963).

Web link: tsourakakis.com/2011/12/29/ahlswede-zhang-identity/

Further reading: *Combinatorics of Finite Sets* by Ian Anderson, Dover reprint, 2003.