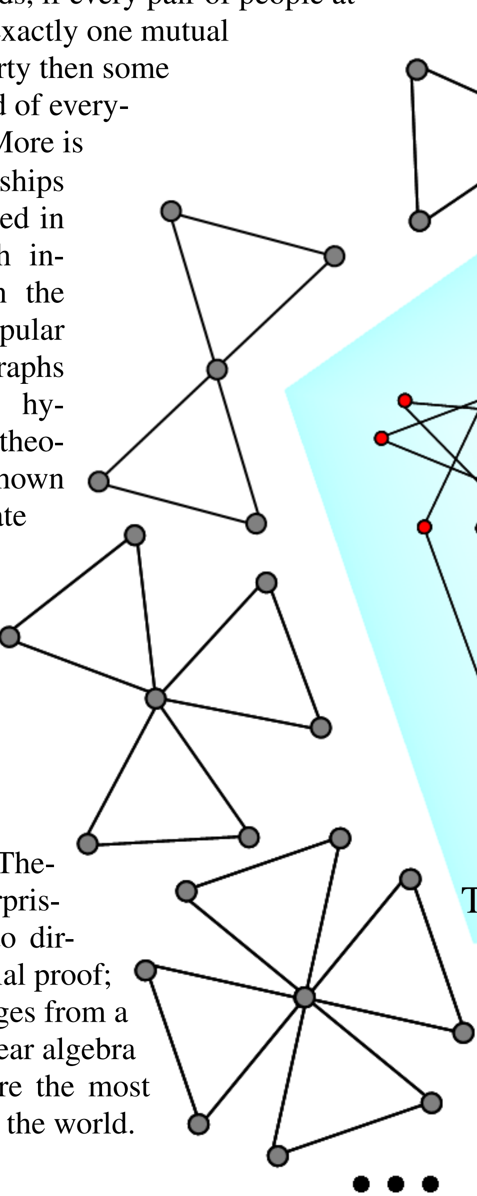
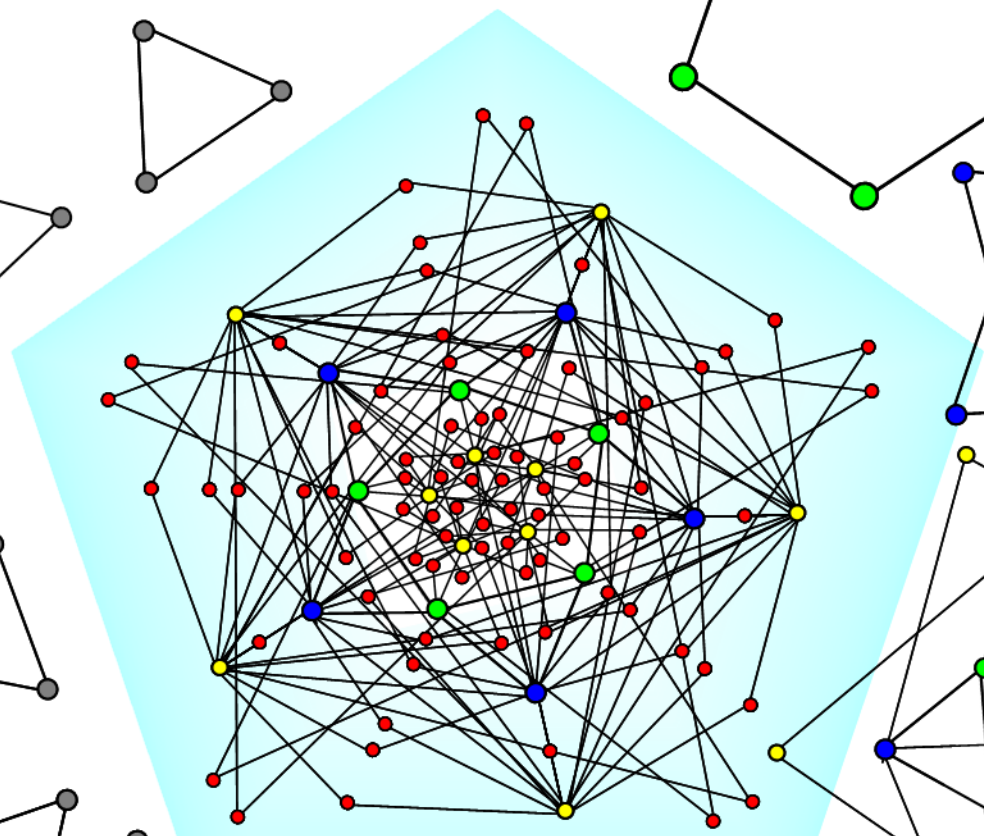
THEOREM OF THE DAY

The Friendship Theorem *In a finite graph in which any two distinct vertices share exactly one common neighbour some vertex is adjacent to all other vertices.*

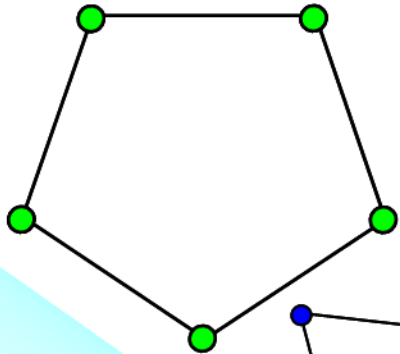
. . . in other words, if every pair of people at a party shares exactly one mutual friend at the party then some guest is a friend of everybody present. More is is true: friendships must be arranged in triangles which intersect only in the universally popular guest. So the graphs satisfying the hypothesis of the theorem are those shown on the immediate right, usually known as 'windmill' graphs.

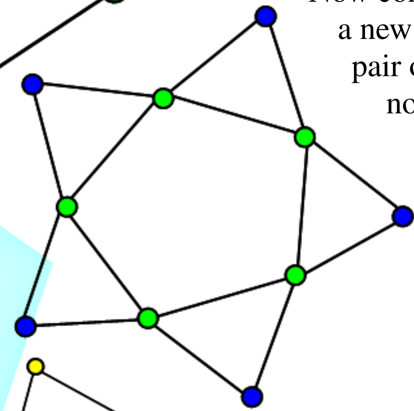


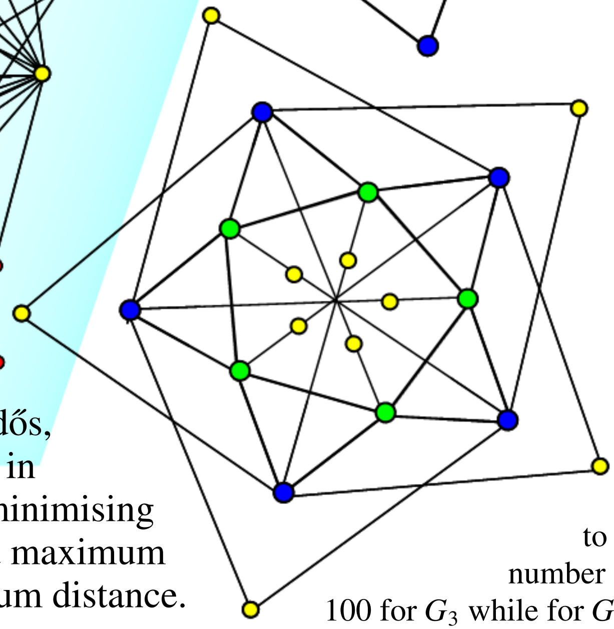
Despite the almost trivial nature of these graphs the Friendship Theorem seems surprisingly resistent to direct combinatorial proof; and yet it emerges from a few lines of linear algebra as though it were the most natural thing in the world.



This theorem was proved by Paul Erdős, Alfréd Rényi and Vera Sós in 1966 in connection with the problem of minimising the edge set of a graph of fixed maximum degree and smallest maximum distance.







Almost more interesting than the theorem itself is the fact that it is false for infinite graphs. Start with $G_0 = C_5$, the pentagon, as shown on the immediate left. Now construct G_{n+1} by creating a new vertex adjacent to each pair of vertices in G_n having no common neighbour. If we repeat this indefinitely the result is an infinite graph satisfying the hypothesis of the theorem (for any pair of vertices we have, by definition, ensured a common neighbour) but not the conclusion (since it is clearly not a windmill graph). The graphs G_0, G_1 and G_2 are shown on the left; G_3 is shown in the centre. **A Challenge:** try to find a formula for the number of vertices in G_n: it is 100 for G_3 while for G_4 the number is 3695.

Web link: the 'book' proof is well described here: math.mit.edu/~fox/MAT307-lecture20.pdf; for the fascinating full story see this MS Dissertation by Katie Leonard: web.pdx.edu/~caughman/Katie.pdf. The Friendship Theorem appears as Theorem 6 near the end of the original paper: 1966-06 at www.renyi.hu/~p_erdos/Erdos.html.

Further reading: *Proofs from the Book,* by Martin Aigner and Günter M. Ziegler, Springer-Verlag, Berlin, 5th Edition, 2014, chapter 43.