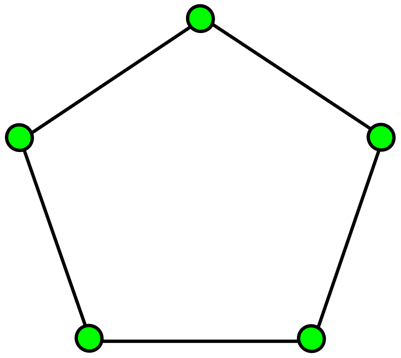
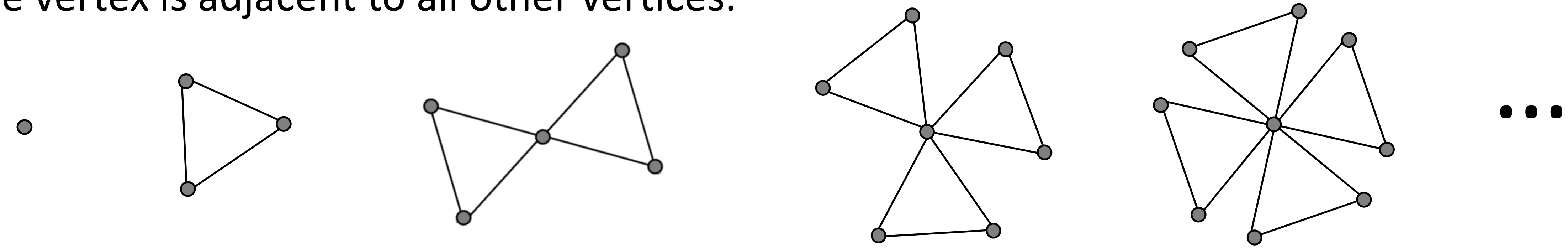
Friendship graphs

Robin Whitty
(with Tony Forbes and Carrie Rutherford)



The Friendship Theorem

In a finite graph in which any two distinct vertices share exactly one common neighbour some vertex is adjacent to all other vertices.



Original proof, using eigenvalue techniques:
Paul Erdős, Alfréd Rényi and Vera T. Sós, "On a problem of graph theory", *Studia Sci. Math. Hungar.*, Vol. 1, 1966, pp. 215–235.

Elementary proof:
Judith Q. Longyear and T.D. Parsons "The friendship theorem“, *Indagationes Mathematicae (Proceedings)*, Vol. 75, Issue 3, 1972, pp. 257–262.

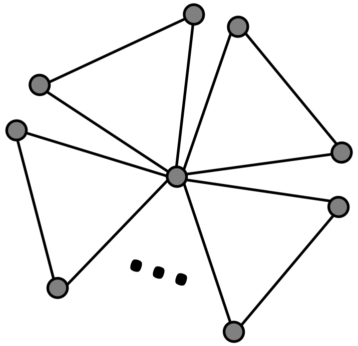
Kotzig’s conjecture (1983, proved 1994, 2000)

There is no finite graph in which every pair of vertices is joined by exactly one path of length $L > 2$.

Friendship graphs

A graph in which any pair of distinct vertices is joined by exactly one path of length two.

Finite friendship graphs are in short supply: they consist of a single vertex which is common to zero or more triangles.



Infinite friendship graphs, however, are very numerous!

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THERE ARE $2^{\aleph_\alpha}$ FRIENDSHIP GRAPHS OF CARDINAL $\aleph_\alpha$

BY

VÁCLAV CHVÁTAL, ANTON KOTZIG, IVO G. ROSENBERG, AND ROY O. DAVIES

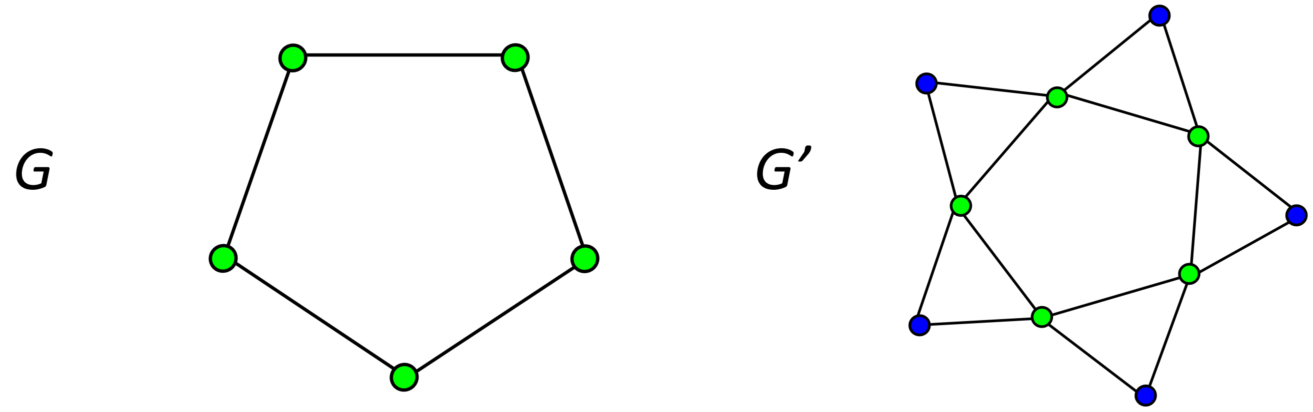
A *friendship graph* is a graph in which every two distinct vertices have exactly one common neighbour. Finite friendship graphs were characterized by Erdős, Rényi, and Sós [1] as those for which the vertices can be enumerated as $u, v_1, \ldots, v_k, w_1, \ldots, w_k$ in such a way that the only edges are uv_i, uw_i, and v_iw_i $(i = 1, \ldots, k)$. Thus finite friendship graphs are rather rare. In contrast, we shall show that there are as many nonisomorphic friendship graphs of given infinite cardinal as there are nonisomorphic graphs of that cardinal altogether. In fact, we do a little more.

THEOREM. *Let c be a cardinal with $3 \leq c \leq \aleph_0$. Then there are $2^{\aleph_\alpha}$ nonisomorphic c-chromatic friendship graphs of cardinal $\aleph_\alpha$.*

This was proved for $\alpha = 0$ in [2], and we use here a similar method. The range of c cannot be extended, because firstly every nontrivial friendship graph contains a triangle and is therefore at least 3-chromatic, and secondly every friendship graph is without cycles of length 4 and therefore is at most $\aleph_0$-chromatic by a result of Hajnal [3, Cor. 5.6].

With every graph G, associate a graph G' by introducing one additional vertex w_{uv} for every two distinct vertices u, v of G such that u and v have no common neighbour in G, and joining w_{uv} to both u and v. Let $G'' = (G')'$ and so on, and ext $G = \bigcup_{i=1}^{\infty} G^{(i)}$. We summarize the principal properties of this extension.

The Chvátal–Kotzig construction



LEMMA 1. *If G is infinite then* ext G *has the same cardinal as G. If G has chromatic number at least 3 then* ext G *has the same chromatic number as G. If G contains no cycle of length 4 then* ext G *is a friendship graph. If H is a finite subgraph of* ext G *and every vertex of H has degree at least 3, then H is in fact a subgraph of G.*

Only the last statement requires proof. Suppose if possible that $H \cap (\text{ext } G \setminus G) \neq \emptyset$, let i be the largest integer such that $H \cap (G^{(i)} \setminus G^{(i-1)})$ is nonempty, and let h be any element of this set. Since h has degree at least 3 in H, and in ext G is joined by at most two edges to points of ext $G \setminus \bigcup_{j=i+1}^{\infty} G^{(j)}$, it is joined in H to a point of $\bigcup_{j=i+1}^{\infty} G^{(j)}$. This contradicts the definition of i.

LEMMA 2. *There exists a finite graph Γ, with two distinguished vertices a and b, such that*

(i) *every vertex of Γ has degree at least 3, and Γ is 3-chromatic and contains no cycle of length 4;*

(ii) *there is a 3-coloring of Γ in which a and b have the same color, but there exists no automorphism of Γ that interchanges a and b;*

(iii) *every path from a to b in Γ has length at least 3.*

Such a graph is exhibited, together with an appropriate coloring, in Figure 1. Verifications are left to the reader.

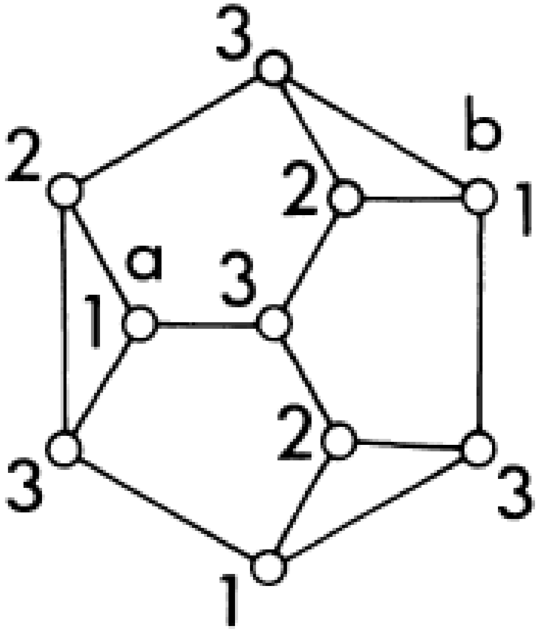
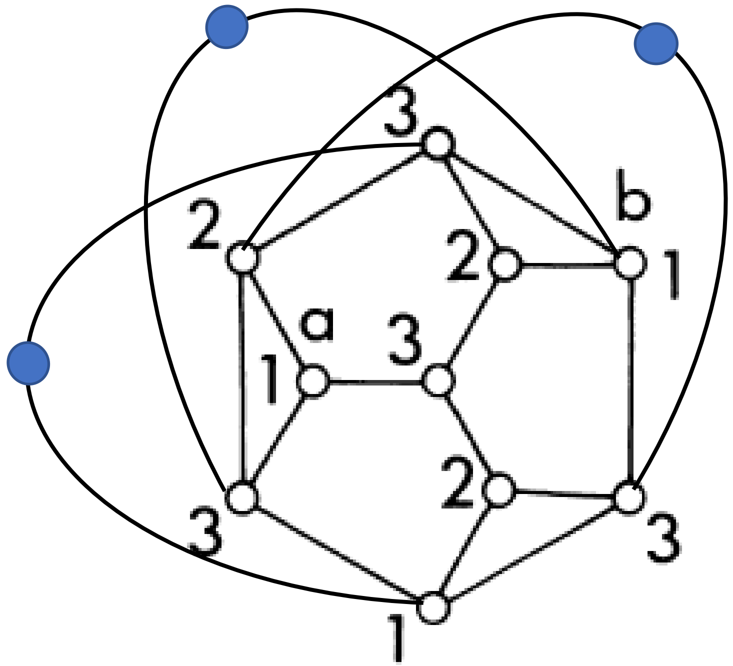


Figure 1



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DEGREES OF VERTICES IN A FRIENDSHIP GRAPH

BY
ANTON KOTZIG

Infinite friendship graphs have been constructed by Chvátal, Kotzig, Rosenberg and Roy O. Davies [1]. The purpose of this paper is to prove the following theorem on degrees of vertices in an infinite friendship graph G:

THEOREM. *Let G be a friendship graph. Then either G contains a vertex which is adjacent to each other vertex of G and then each other vertex of G is of degree two or G does not contain any such vertex and then each vertex of G is of infinite degree.*

LEMMA 1. *The smallest friendship graph is isomorphic to a triangle. Let G be a friendship graph with $|V(G)|>3$. Then (i) G is of diameter two; (ii) G does not contain any circuit of length four; (iii) each edge of G belongs to exactly one triangle. (=circuit of length three).*

COROLLARY 1. *A friendship graph G is uniquely decomposible into triangles. If G contains a vertex v of finite degree, then* $d_G(v) \equiv 0 (mod\, 2)$.

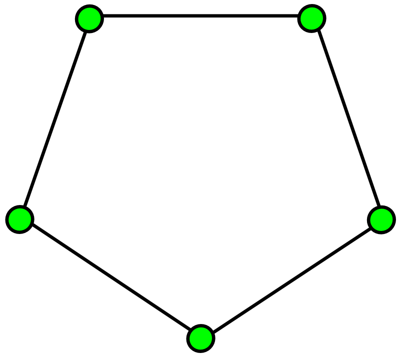
COROLLARY 2. *The neighbourhood* $N_v = F$ *of each vertex v of a friendship graph G is a* 1*-regular graph (in sense of Harary* [3], *because we have for each* $w \in V(F)$ *that the vertices v and w have exactly* $d_F(w)$ *common neighbours in G; thus* $d_F(w) = 1$ *for each w of* $V(F)$).

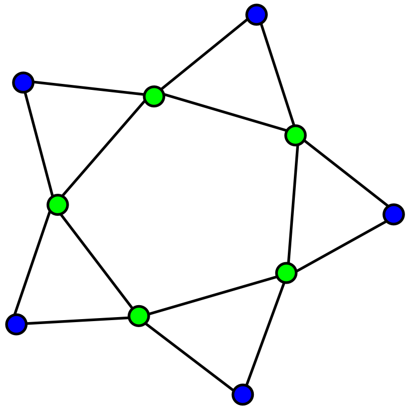
LEMMA 2. *If G is an infinite friendship graph and x a vertex of G with* $2 < d_G(x) = 2n < \infty$, *then each neighbour of x is of infinite degree.*

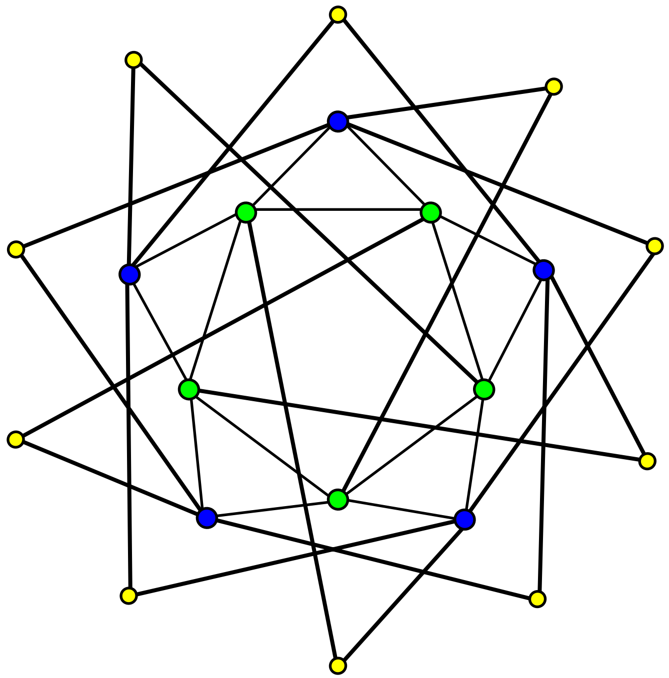
LEMMA 3. *Let x and v be two adjacent vertices both of infinite degree in a friendship graph G. Then each neighbour of x (or of v, respectively) is of infinite degree.*

COROLLARY 3. *A vertex of an infinite friendship graph is either of degree two or of infinite degree.* {*If we suppose that the vertex x of an infinite friendship graph is of degree* $2 < d_G(x) < \infty$, *then we have* $d_G(u) = \infty = d_G(v)$ *for each edge* $[u, v]$ *belonging to a triangle which contains x, where* $u \neq x \neq v$ *(see Lemma* 2). *But then (according to Lemma* 3) *x must be of infinite degree, which is a contradiction of our supposition. Thus x cannot be of a finite degree greater than two*}.

Ext G when G is a pentagon







...

Question: how many vertices in the n-th iteration?

Tony Forbes

to me, Carrie

Thu, Dec 2

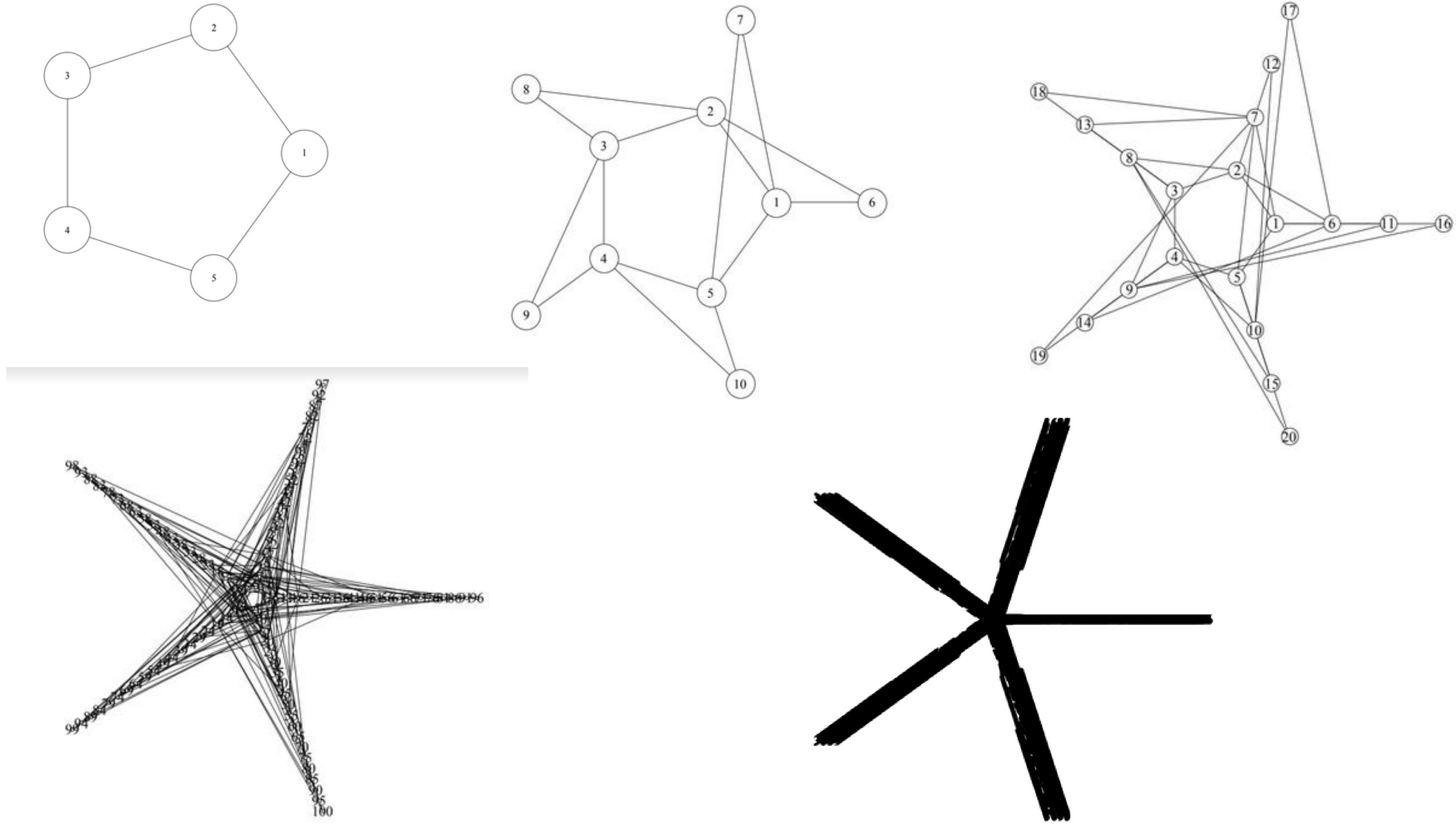
I created generations 1-4 of the infinite friendship graph.

(i) I agree: generation 4 has 3695 vertices.

(ii) Generation 4 also has 6950 adjacent pairs with no common neighbour and 6528225 non-adjacent pairs with no common neighbour.

(iii) Hence generation 5 has 3695 + 6950 + 6528225 = 6538870 vertices.

(iv) My computer has run out of steam; so I cannot create generation 5.



1 2 3 4 5
1 2 3 4 5 6 7 8 9 10
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20
96 97 98 99 100

This is all I have for generations 0 to 5. Find out what n_0 should be in terms of previous generations and the problem is solved.

{vertices, edges, degree frequencies, number of adjacent pairs with 0 common neighbours, number of adjacent pairs with 1 common neighbour, number of non-adjacent pairs with 0 common neighbours, number of non-adjacent pairs with 1 common neighbour}

Merry Christmas,

T.

TraditionalForm=

5	5	$\begin{pmatrix} 2 & 5 \end{pmatrix}$	5	0	0	5
10	15	$\begin{pmatrix} 2 & 5 \\ 4 & 5 \end{pmatrix}$	0	15	10	20
20	35	$\begin{pmatrix} 2 & 10 \\ 5 & 10 \end{pmatrix}$	20	15	60	95
100	195	$\begin{pmatrix} 2 & 80 \\ 7 & 5 \\ 13 & 15 \end{pmatrix}$	120	75	3475	1280
3695	7385	$\begin{pmatrix} 2 & 3595 \\ 56 & 10 \\ 77 & 75 \\ 83 & 15 \end{pmatrix}$	6950	435	6 528 225	289 055
6 538 870	13 077 735	$\begin{pmatrix} 2 & 6535175 \\ ? & ? \\ ? & ? \\ ? & ? \\ ? & ? \end{pmatrix}$	13 056 450	21 285	n_0	$21\,378\,394\,091\,280 - n_0$

Rutherford, Carrie
to me, Tony

Dec

Just stating the obvious:

- no. vertices = [no. verts + no. adjacent pairs with 0 common neighbours + no. non-adjacent pairs with 0 common neighbours] in previous generation
- no. verts w/ degree 2 = [no. adjacent pairs with 0 common neighbours + no. non-adjacent pairs with 0 common neighbours] in previous generation
- no. edges = [no. edges + 2*(no. adjacent pairs with 0 common neighbours + no. non-adjacent pairs with 0 common neighbours)] in previous generation

Not so obvious (or maybe I'm just being thick!):

- no. adjacent pairs with 0 common neighbours = [2*(no. non-adjacent pairs with 0 common neighbours)] in previous generation

Tony Forbes
to Carrie, me

Here's another observation, which I don't immediately see how to prove (possibly due to thickness):

#(half-edges attached to vertices of degree > 2) - #(half-edges attached to vertices of degree 2)
= 2 #(edges in the previous generation).

E.g. 56*10 + 77*75 + 83*15 - 2*3595 = 2*195.

How to generate the ext graphs by rotational symmetry