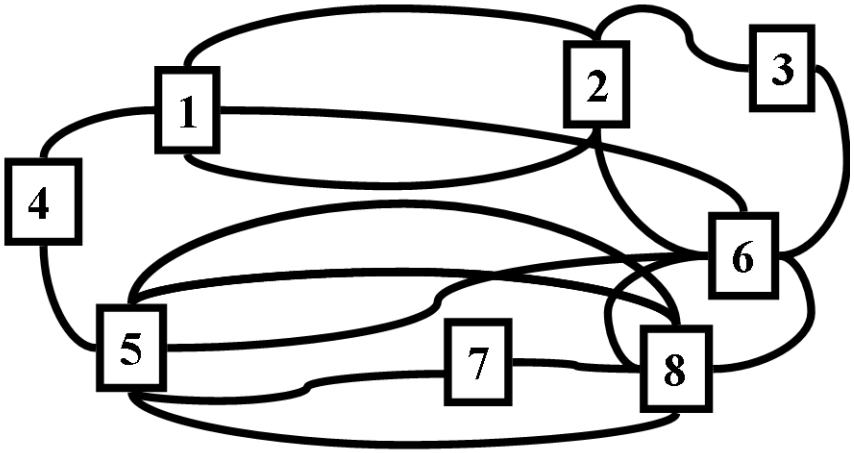
THEOREM OF THE DAY

The Euler–Hierholzer "Bridges of Königsberg" Theorem *A connected graph G has an Euler tour if and only if every vertex has even degree.*



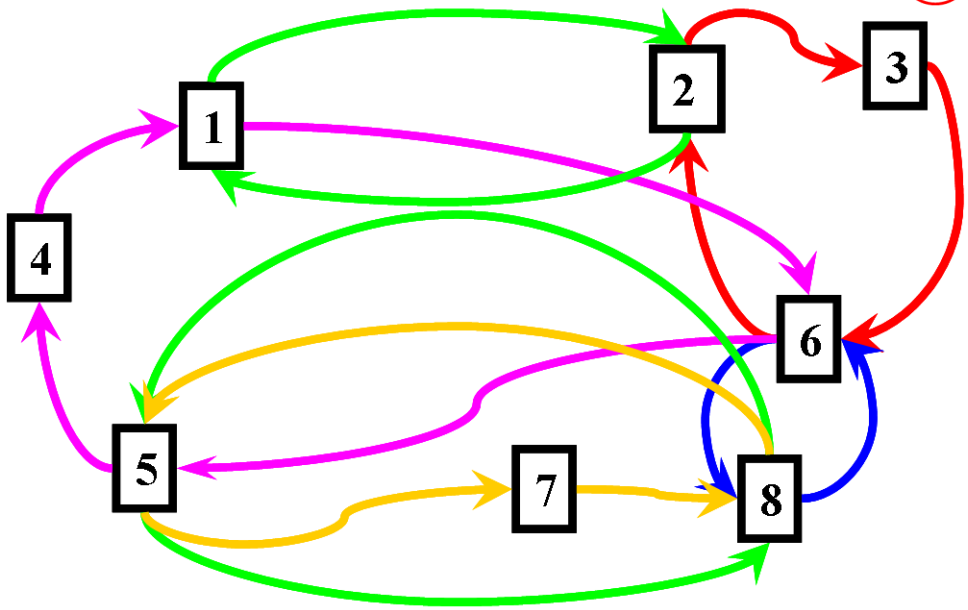
α = (1 2)(5 8)

β =(1 6 5 4)

γ = (2 3 6)

δ=(5 7 8)

ε = (6 8)



The problem of constructing an *Euler tour* can be paraphrased as: draw round all the edges of the graph and return to the beginning without lifting your pen and without drawing over any edge twice. More formally, we seek a sequence of consecutive edges which begins and ends at the same vertex and traverses every edge exactly once. Any solution will necessarily use up pairs of entries and exits to and from each vertex: hence the necessity of even degree.

A completed tour will partition the edges into disjoint cycles: sub-tours without repeated vertices. Any other tour can then be constructed by jumping from cycle to cycle. You can think of the circuits, taken separately or together, as permutations of the vertices involved. Five have been suggested here: $\alpha, \beta, \gamma, \delta$ and ε. An Euler tour beginning and ending at vertex 1 will consist of a product of powers of these permutations which leaves element 1 fixed. For instance, the path

$$1 \xrightarrow{\alpha} 2 \xrightarrow{\gamma} 3 \xrightarrow{\gamma} 6 \xrightarrow{\beta^{-1}} 1 \xrightarrow{\alpha^{-1}} 2 \xrightarrow{\gamma^{-1}} 6 \xrightarrow{\beta} 5 \xrightarrow{\delta^{-1}} 8 \xrightarrow{\varepsilon} 6 \xrightarrow{\varepsilon} 8 \xrightarrow{\alpha^{-1}} 5 \xrightarrow{\delta} 7 \xrightarrow{\delta} 8 \xrightarrow{\alpha} 5 \xrightarrow{\beta} 4 \xrightarrow{\beta} 1$$

corresponds to the product of permutations $\alpha\gamma^2\beta^{-1}\alpha^{-1}\gamma^{-1}\beta\delta^{-1}\varepsilon^2\alpha^{-1}\delta^2\alpha\beta^2$ which evaluates to (2 5 6)(3 4 8).

Puzzle: find an Euler tour of the above graph whose product of permutations fixes all vertices, if such a tour exists.

Euler discovered his necessary condition (the 'only if' part of the theorem) in 1736 as a solution to the famous "Bridges of Königsberg" problem, foreshadowing thereby the study of topology and graph theory. It was proved sufficient in 1873 by Carl Hierholzer.

Web link: plus.maths.org/content/bridges-k-nigsberg

Further reading: *Graph Theory: 1736–1936* by Norman L. Biggs, E. Keith Lloyd and Robin. J. Wilson, Clarendon Press, 1986.