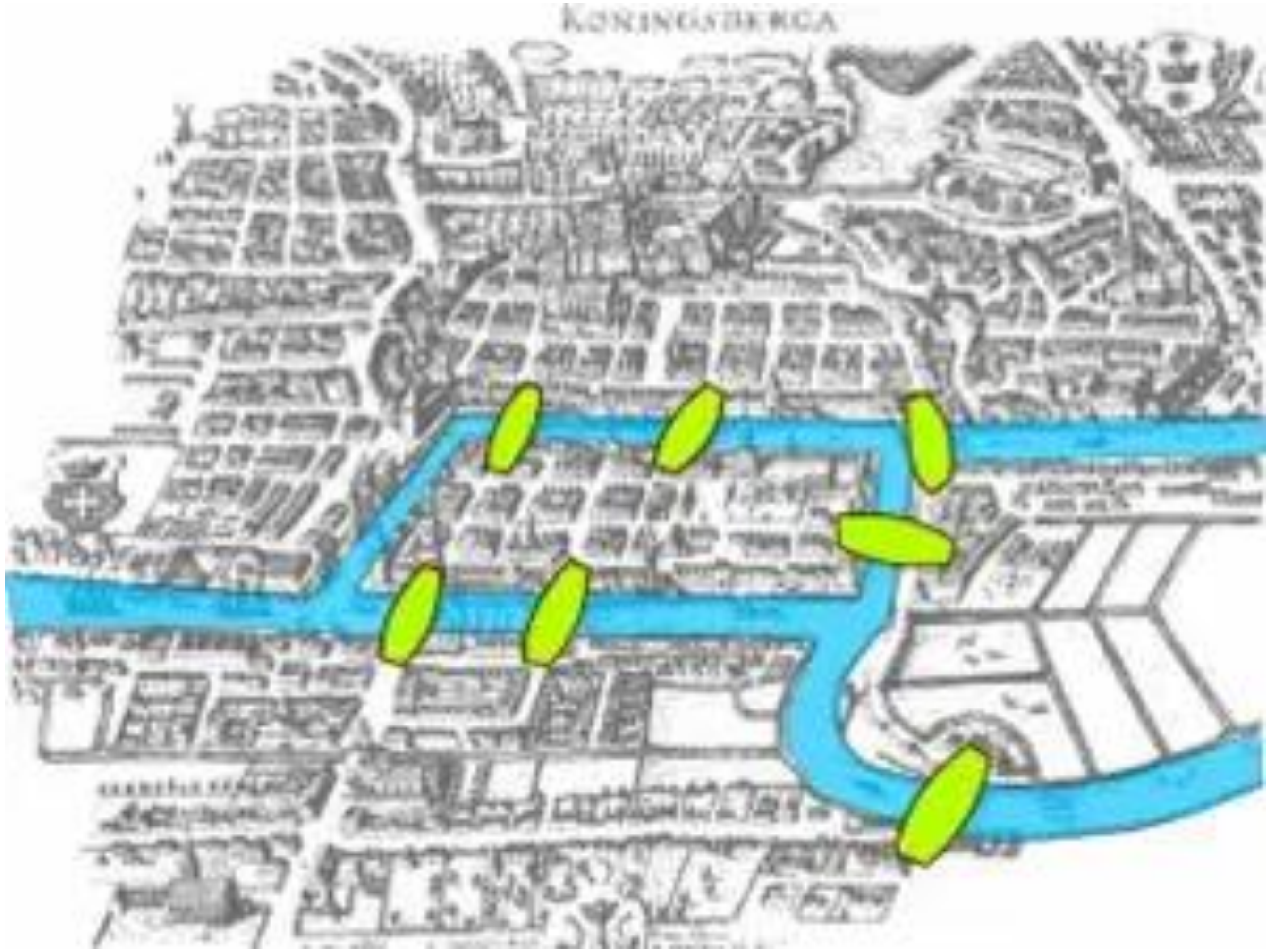
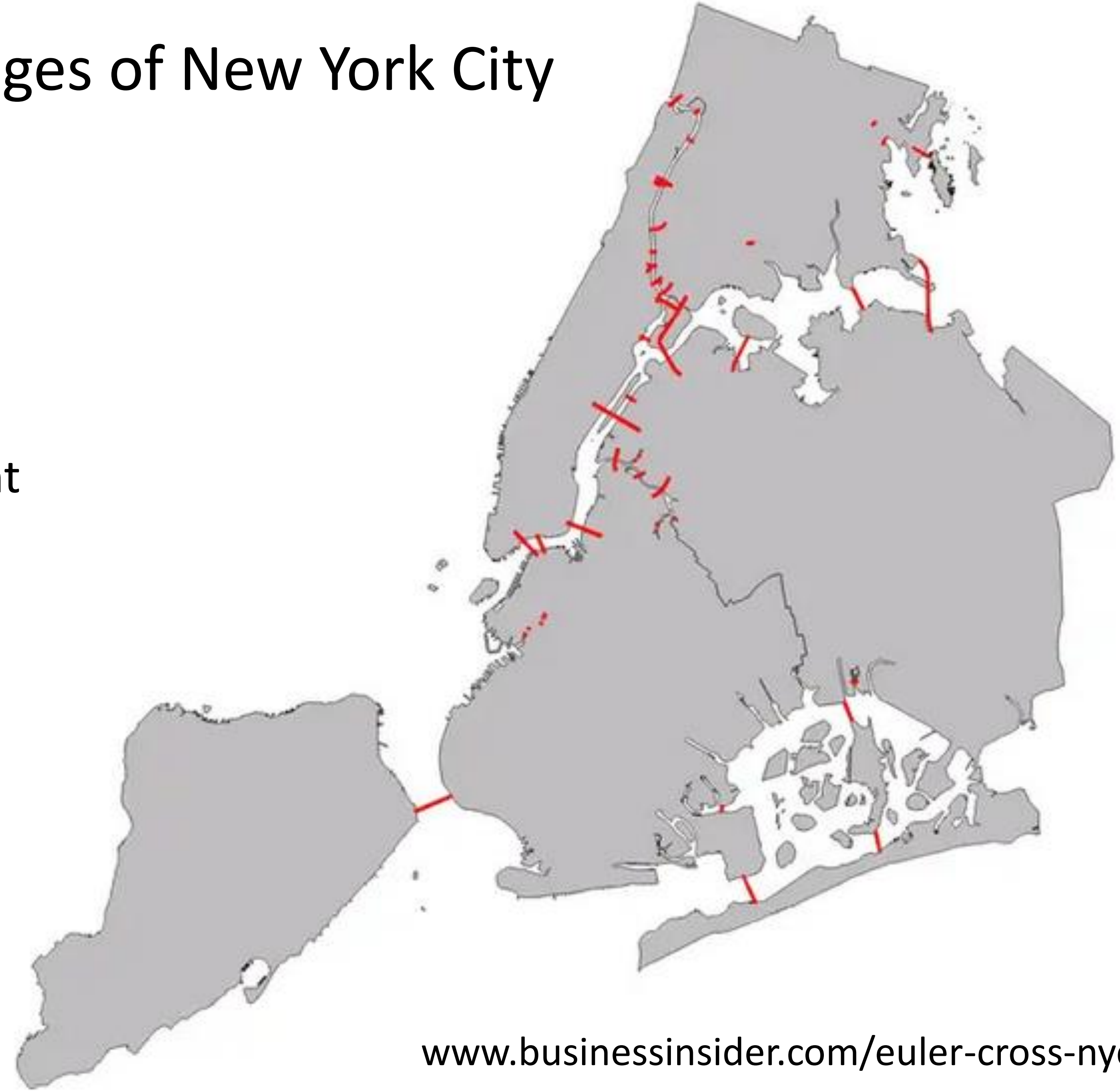
Euler tours and sets of permutations



Robin Whitty,
MSG, February 2023

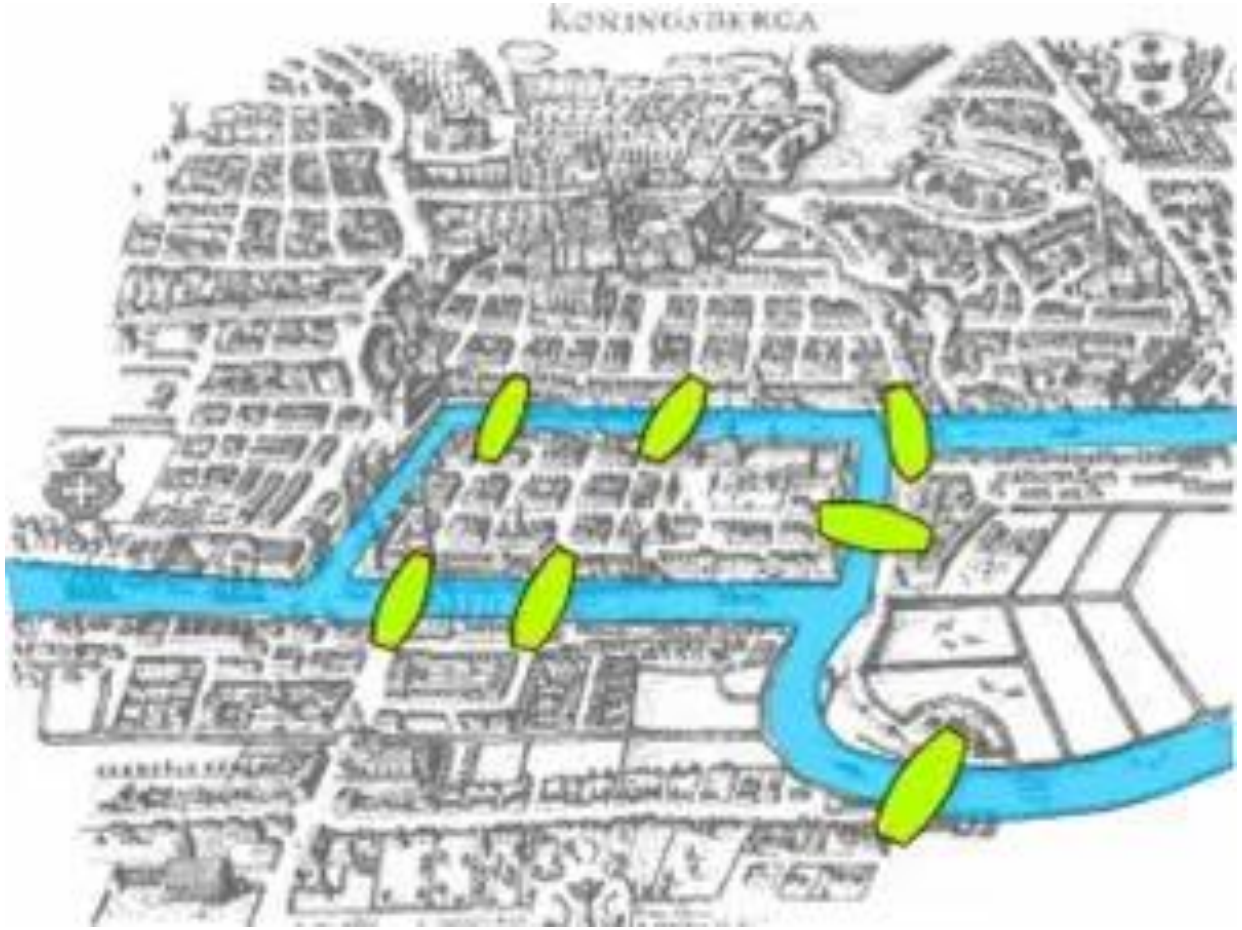
Walking the bridges of New York City

Can you walk around NYC in such a way as to cross each bridge exactly once before returning to your point of departure?

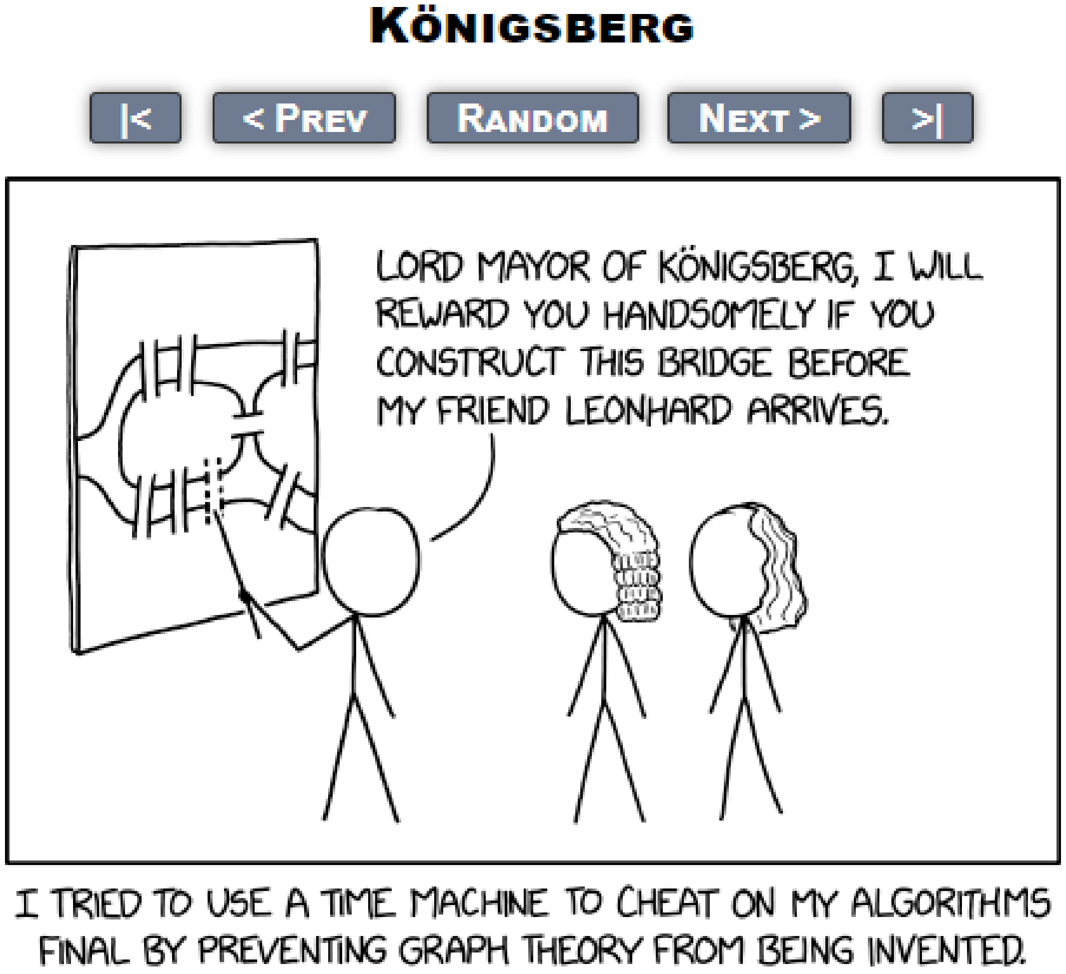


www.businessinsider.com/euler-cross-nyc-bridges-once-2014-1

Euler’s 1736 answer for Königsberg



NO



xkcd.com/2694/

In graph-theoretic terms I

Multiple edges: distinct edges having the same two endpoints;

Consecutive edges: $e = xy$ and $e' = x'y'$ are **consecutive** if $y = x'$;

Walk: sequence of consecutive edges, the number of edges in the sequence being the **length** of the walk;

Connected graph: graph in which any two vertices have a walk joining them;

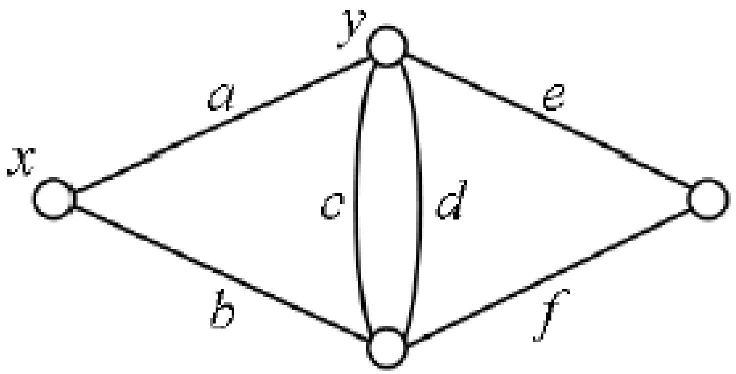
Closed walk: walk whose first and last vertices are identical;

Trail: walk with no repeated edges;

Closed trail: trail whose first and last vertices are identical;

Path: trail with no repeated vertices, except perhaps first and last, in which case we have:

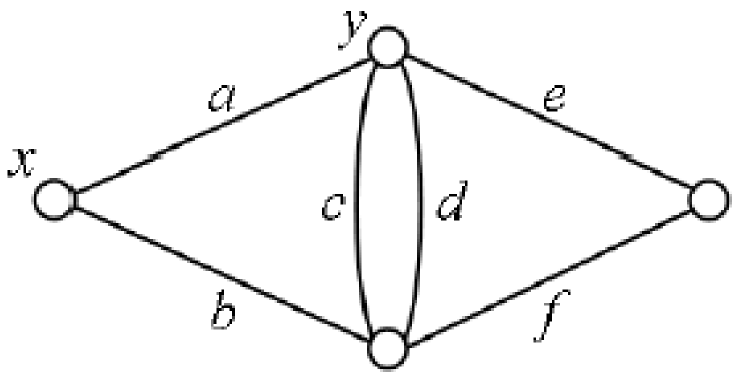
Cycle: path whose first and last vertices are identical.



length	walks	trails	paths
2	bc, bd	bc, bd	bc, bd
3	bfe, acd, adc, acc, add, aee, bba, aaa	bfe, acd, adc	bfe
4	$acfe$, $adfe$, $aefc$, $aefd$, $aabc$, and 19 others	$acfe$, $adfe$, $aefc$, $aefd$	none

In graph-theoretic terms II

Definition In a graph G an **Euler tour** is a closed trail containing every edge. A graph is called **Eulerian** if it admits (i.e. 'has') an Euler tour.



Example: we give some Euler tours of the above graph:

(1) $a\,c\,d\,e\,f\,b$

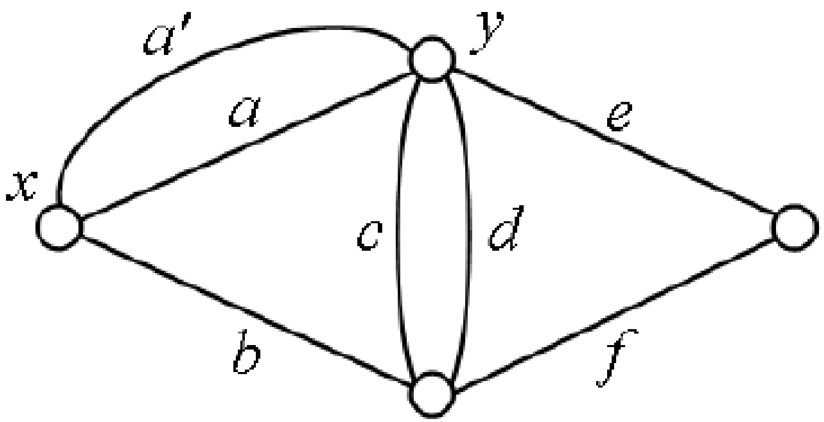
(2) $b\,f\,e\,d\,c\,a$
(which is (1) reversed)

(3) $a\,c\,f\,e\,d\,b$
(which is (1) with the cycle *def* reversed)

(4) $c\,d\,e\,f\,b\,a$
(which is (1) but 'cycled round' to start at vertex *y*)

The Euler-Hierholzer Theorem

Here is the example graph from the last lecture but with an extra edge a' added:



It is not hard to see that an Euler tour is no longer possible: indeed, suppose there *was* a tour, S, say. We can assume, without loss of generality[1] that S starts with edge a (if it starts with b and ends with a we can just reverse the tour; and a and a' are copies of the same edge). If S ends with a' then how can edge b be included in the tour? If S ends with b then how can a' be included? We conclude that the number of edges at x is incompatible with an Euler tour.

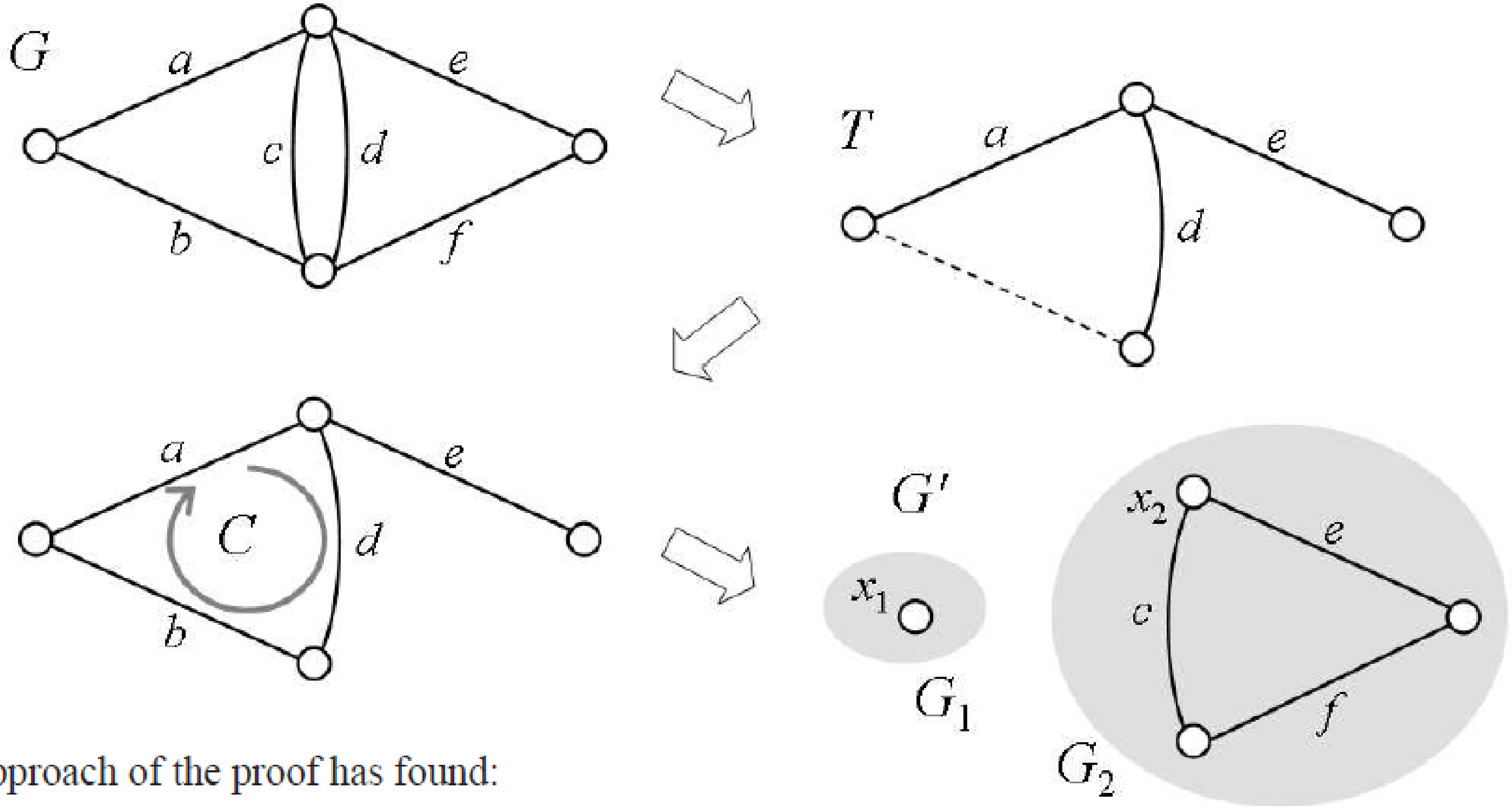
The argument above that a vertex of degree 3 does not allow an Euler tour can be extended to a necessary and sufficient condition:

The Euler-Hierholzer Theorem A graph G is Eulerian if and only if it is connected and every vertex of G has even degree.

(Euler proved 'if' in 1736. **Carl Hierholzer** (1840 –1871) proved 'only if', algorithmically)

Proof of Euler – Hierholzer, only-if

Without loss of generality assume G is our example graph…



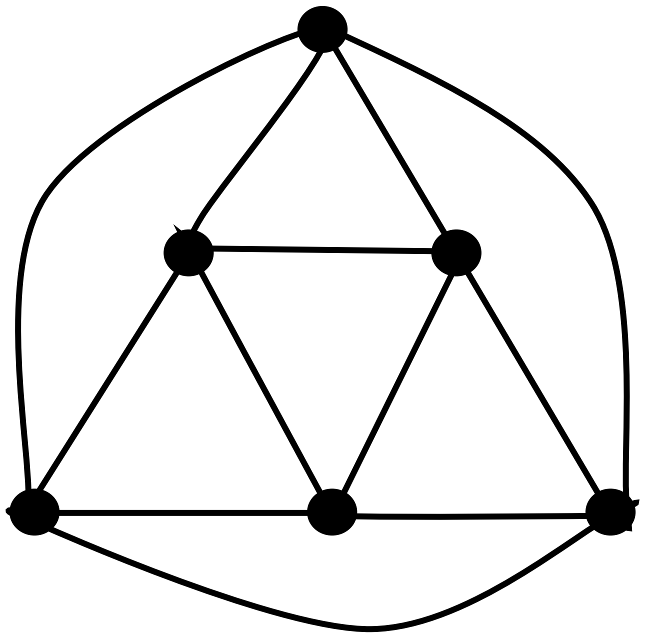
The approach of the proof has found:

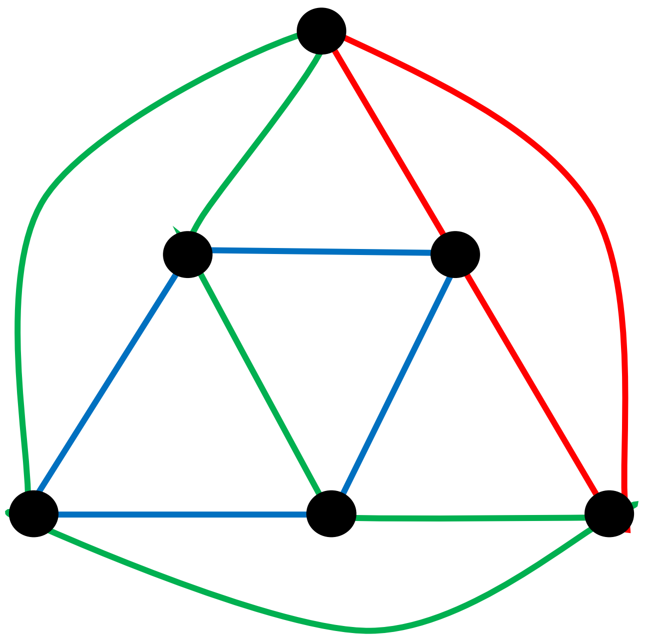
- a spanning tree T (edges a, d, e);
- an edge leaving a leaf (edge b);
- the fundamental cycle formed by b (that is $C = adb$); and
- two components formed by deleting the edges of C (component G_1 is a single vertex and has no edges; G_2 is the cycle cef).

Easy corollary

A graph *G* has an Euler tour if and only if its edge set may be expressed as a union of disjoint cycles of *G.*

If: each vertex *v* lies on a number of cycles each of which arrive at and leave *v.* Therefore *v* has even degree.
Only if: write down the Euler tour. Each time it repeats a vertex a cycle has been created.

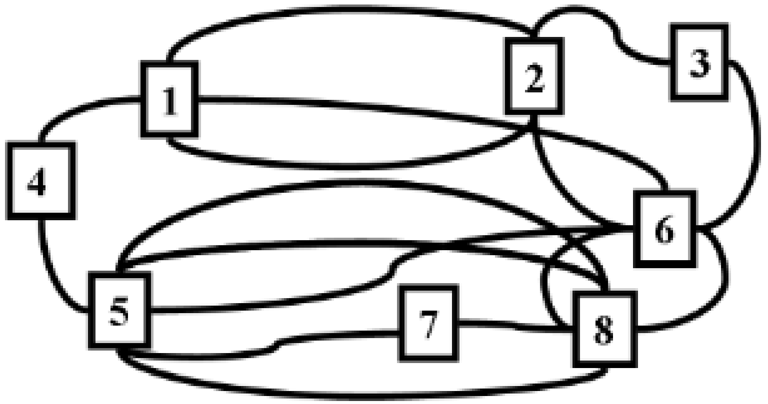




Euler tours and permutations I

THEOREM OF THE DAY

The Euler–Hierholzer "Bridges of Königsberg" Theorem *A connected graph G has an Euler tour if and only if every vertex has even degree.*



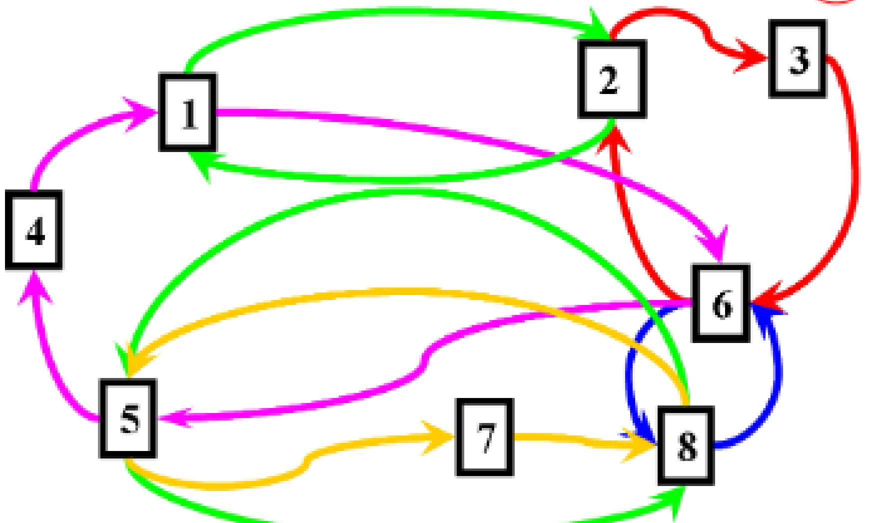
$\alpha = (1\ 2)(5\ 8)$

$\beta = (1\ 6\ 5\ 4)$

$\gamma = (2\ 3\ 6)$

$\delta = (5\ 7\ 8)$

$\varepsilon = (6\ 8)$



The problem of constructing an *Euler tour* can be paraphrased as: draw round all the edges of the graph and return to the beginning without lifting your pen and without drawing over any edge twice. More formally, we seek a sequence of consecutive edges which begins and ends at the same vertex and traverses every edge exactly once. Any solution will necessarily use up pairs of entries and exits to and from each vertex: hence the necessity of even degree.

A completed tour will partition the edges into disjoint cycles: sub-tours without repeated vertices. Any other tour can then be constructed by jumping from cycle to cycle. You can think of the circuits, taken separately or together, as permutations of the vertices involved. Five have been suggested here: $\alpha, \beta, \gamma, \delta$ and ε. An Euler tour beginning and ending at vertex 1 will consist of a product of powers of these permutations which leaves element 1 fixed. For instance, the path

$$1 \xrightarrow{\alpha} 2 \xrightarrow{\gamma} 3 \xrightarrow{\gamma} 6 \xrightarrow{\beta^{-1}} 1 \xrightarrow{\alpha^{-1}} 2 \xrightarrow{\gamma^{-1}} 6 \xrightarrow{\beta} 5 \xrightarrow{\delta^{-1}} 8 \xrightarrow{\varepsilon} 6 \xrightarrow{\varepsilon} 8 \xrightarrow{\alpha^{-1}} 5 \xrightarrow{\delta} 7 \xrightarrow{\delta} 8 \xrightarrow{\alpha} 5 \xrightarrow{\beta} 4 \xrightarrow{\beta} 1$$

corresponds to the product of permutations $\alpha\gamma^2\beta^{-1}\alpha^{-1}\gamma^{-1}\beta\delta^{-1}\varepsilon^2\alpha^{-1}\delta^2\alpha\beta^2$ which evaluates to (2 5 6)(3 4 8).

Puzzle: find an Euler tour of the above graph whose product of permutations fixes all vertices, if such a tour exists.

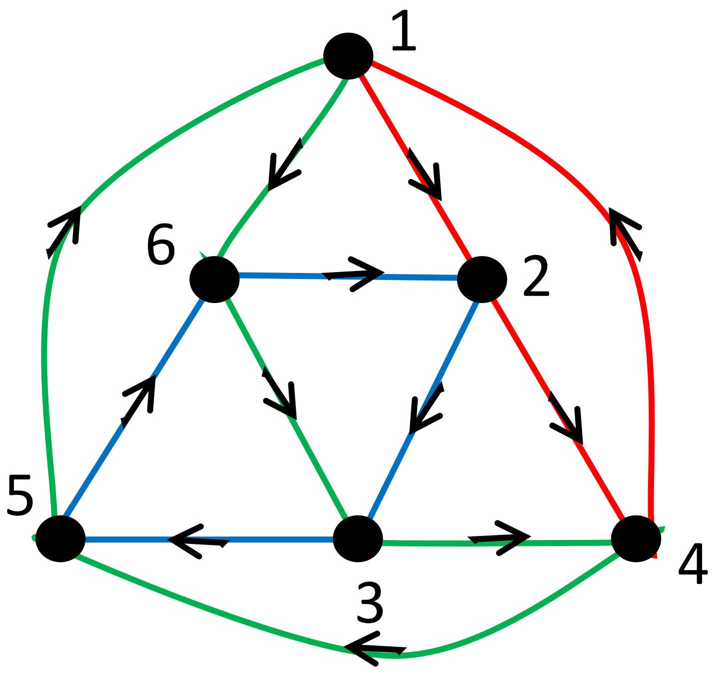
Euler discovered his necessary condition (the 'only if' part of the theorem) in 1736 as a solution to the famous "Bridges of Königsberg" problem, foreshadowing thereby the study of topology and graph theory. It was proved sufficient in 1873 by Carl Hierholzer.

Web link: plus.maths.org/content/bridges-k-nigsberg. See www.maa.org/programs/maa-awards/writing-awards/the-truth-about-konigsberg for historical background.

Further reading: *Graph Theory: 1736–1936* by Norman L. Biggs, E. Keith Lloyd and Robin. J. Wilson, Clarendon Press, 1986.

Euler tours and permutations II

Take an Euler tour disjoint cycle decomposition.
Orient each cycle arbitrarily.
Write the corresponding permutation of the vertices of each cycle.



Now any Euler tour can be written in terms of products of powers of permutations, the powers being the number of edges used as a decomposition cycle is used.

For this example, suppose we take the Euler tour
1 4 2 3 5 4 3 6 5 1 6 2 1.
The resulting permutation products must give a permutation which fixes 1.

```
> a:=[[1,2,4]]:
  b:=[[1,6,3,4,5]]:
  c:=[[2,3,5,6]]:
> pprod([pow(a,-2),pow(c,2),pow(b,-3),pow(c,-1),pow(b,2),pow(c,1),pow(a,-1)]);
                                    [[2, 5], [3, 4]]
```

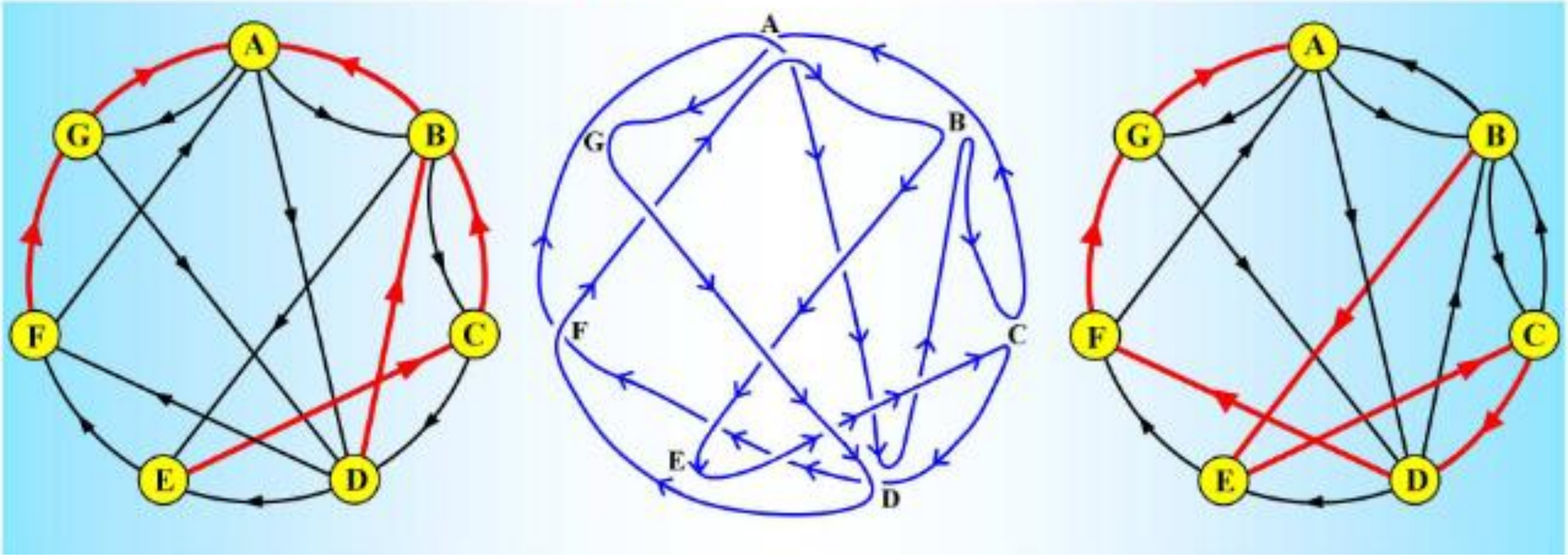
How to explore the space of permutation products

THEOREM OF THE DAY

The BEST Theorem *Let $G = (V, E)$ be a directed graph in which, for each vertex v in V, the indegree and outdegree have the same value, d(v), say. Then G has a directed Euler tour: a closed walk which passes each edge exactly once; let ε(G) denote the number of such tours. Then, for any fixed vertex x,*

$$\varepsilon(G) = t_x \prod_{v \in V} (d(v) - 1)!$$

where t_x denotes the number of those spanning trees of G in which every vertex has a directed path to x.



Counting spanning trees

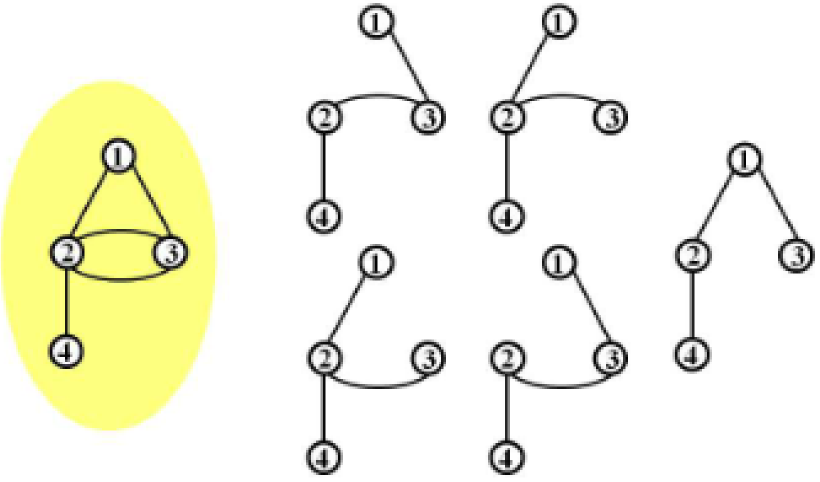
We need the directed graph version of this. Moreover by replacing the edge entries by indeterminates the determinant value becomes a multi-variate polynomial whose terms correspond to individual spanning trees.

THEOREM OF THE DAY

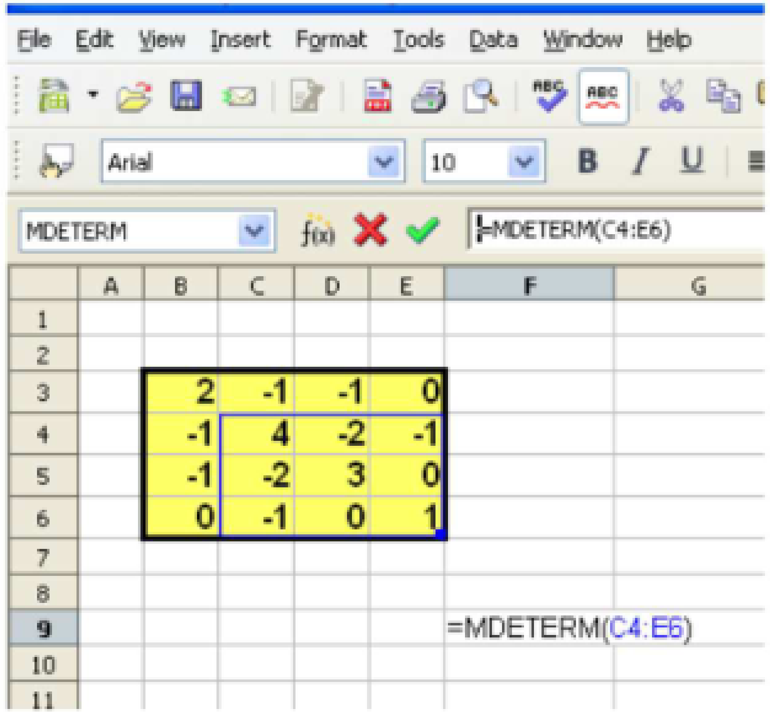
The Matrix Tree Theorem *Let G be a graph with n vertices and let $L(G)$ be the $n \times n$ matrix whose entry in row i and column j is defined to be*

$$\begin{cases} -(\text{the number of edges joining vertex } i \text{ to vertex } j) & \text{if } i \neq j, \quad \text{and} \\ \text{the number of edges incident with vertex } i & \text{if } i = j. \end{cases}$$

Then the number of spanning trees of G is given by $\det L(G)(1|1)$, *where $L(G)(1|1)$ is the matrix obtained by deleting the 1st row and 1st column of $L(G)$.*



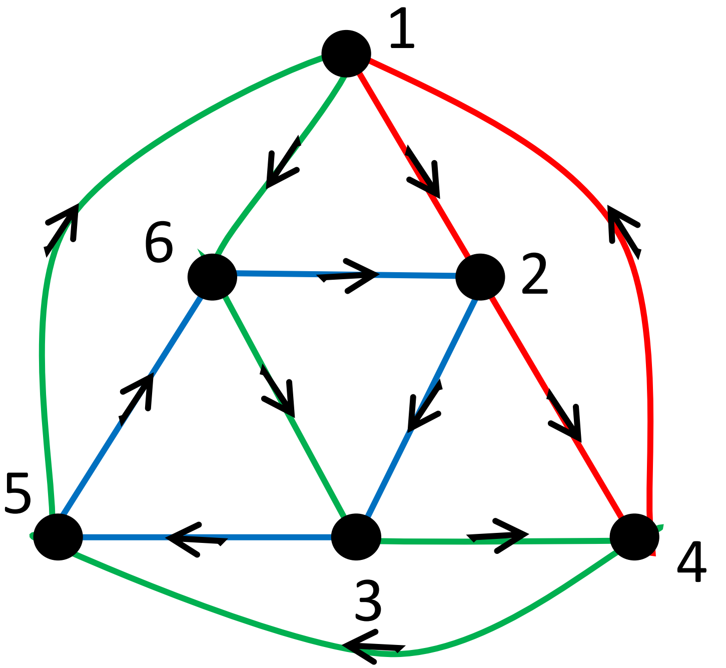
A 4-vertex, 5-edge graph is shown above left. Its spanning trees, shown in the centre, are those connected subsets of edges which are incident with all 5 vertices and which contain no cyclic paths. They necessarily have 4 − 1 = 3 edges. Notice that these spanning trees are distinct as **labelled** objects: there are only two 'structurally' different trees.



An Application
How 'reliable' is the graph on the left? We could interpret this to mean, say, what is the probability that deleting two edges at random from the graph on the left will still leave a connected graph? Removing two edges gives $\binom{5}{2} = 10$ subgraphs on three edges. Of these, the only subgraphs that are connected are the five spanning trees. So our probability is $\frac{5}{10} = \frac{1}{2}$.

The *determinant* function, det, yields a single figure from a square matrix or table. It is available in standard spreadsheet applications as the '=MDETERM' function. As shown here in the OpenOffice Calc package, the calculation will produce the answer 5, since there are exactly 5 spanning trees for the given graph. In fact, any row of $L(G)$, not just the first, and any column, may be deleted in the statement of the theorem without changing the absolute value of the result.

Our earlier example oriented graph




```
> n:=6: u:=[1,2,1,6,2,3,2,4,3,4,3,5,4,1,4,5,5,1,5,6,6,2,6,3]:
  g:=QuickDiGraph(u,n):
  e:=OutEdgeSets(g):
  evalm(g),print(e);
```

[[2, 6] [3, 4] [4, 5] [1, 5] [1, 6] [2, 3]]

0	1	0	0	0	1
0	0	1	1	0	0
0	0	0	1	1	0
1	0	0	0	1	0
1	0	0	0	0	1
0	1	1	0	0	0

=

```
> t:=Trees(u,n);
```

t := [[[2, 3], [3, 4], [4, 1], [5, 1], [6, 2]], [[2, 3], [3, 4], [4, 1], [5, 1], [6, 3]], [[2, 3], [3, 4], [4, 1], [5, 6], [6, 2]], [[2, 3], [3, 4], [4, 1], [5, 6], [6, 3]], [[2, 3], [3, 4], [4, 5], [5, 1], [6, 2]], [[2, 3], [3, 4], [4, 5], [5, 1], [6, 3]], [[2, 3], [3, 5], [4, 1], [5, 1], [6, 2]], [[2, 3], [3, 5], [4, 1], [5, 1], [6, 3]], [[2, 3], [3, 5], [4, 5], [5, 1], [6, 2]], [[2, 3], [3, 5], [4, 5], [5, 1], [6, 3]], [[2, 4], [3, 4], [4, 1], [5, 1], [6, 2]], [[2, 4], [3, 4], [4, 1], [5, 1], [6, 3]], [[2, 4], [3, 4], [4, 1], [5, 6], [6, 2]], [[2, 4], [3, 4], [4, 1], [5, 6], [6, 3]], [[2, 4], [3, 4], [4, 5], [5, 1], [6, 2]], [[2, 4], [3, 4], [4, 5], [5, 1], [6, 3]], [[2, 4], [3, 5], [4, 1], [5, 1], [6, 2]], [[2, 4], [3, 5], [4, 1], [5, 1], [6, 3]], [[2, 4], [3, 5], [4, 1], [5, 6], [6, 2]], [[2, 4], [3, 5], [4, 5], [5, 1], [6, 2]], [[2, 4], [3, 5], [4, 5], [5, 1], [6, 3]]]

=

```
> t[1];
```

[[2, 3], [3, 4], [4, 1], [5, 1], [6, 2]]

=

```
> tours:=TourSet(g,t[1],n);
```

tours := [[[2, 6], [4, 3], [5, 4], [5, 1], [6, 1], [3, 2]], [[6, 2], [4, 3], [5, 4], [5, 1], [6, 1], [3, 2]]]

Tours produced according to the (algorithm underlying the) BEST Theorem

=

```
> ET:=ListTourEdges(tours[1],n);
```

ET := [[1, 2], [2, 4], [4, 5], [5, 6], [6, 3], [3, 5], [5, 1], [1, 6], [6, 2], [2, 3], [3, 4], [4, 1]]

=