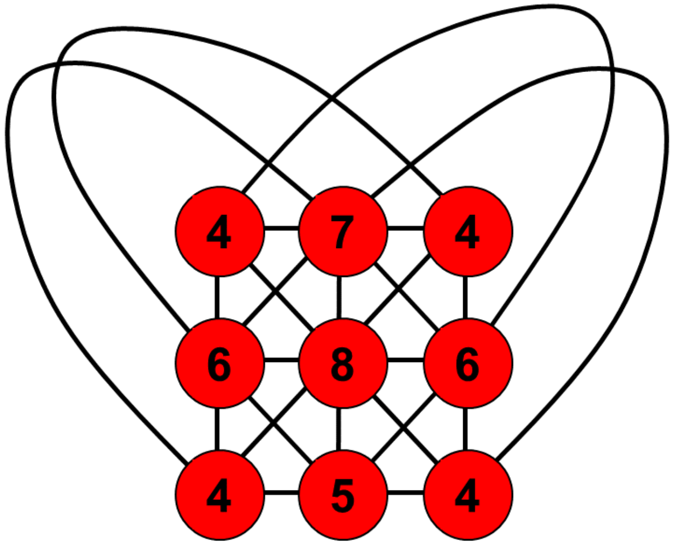
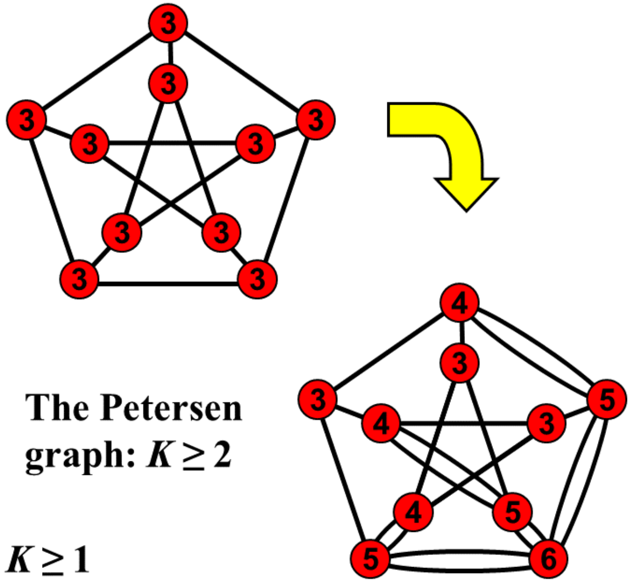
THEOREM OF THE DAY

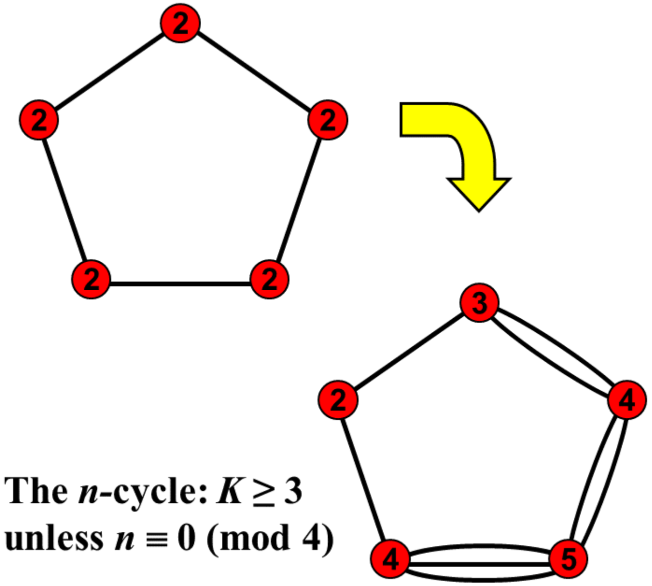
The 1-2-3 Conjecture *Call a graph, G,* degree colourable *if no two adjacent vertices have the same degree. If, by adjusting edge multiplicities in G so that no edge multiplicity exceeds k, we can create a degree colourable graph, then say that G is k*-degree colourable. *Now there exists a positive integer constant K such that any connected graph on at least three vertices is K-degree colourable.*



A graph based on the affine plane of order 3: $K \geq 1$



The Petersen graph: $K \geq 2$



The *n*-cycle: $K \geq 3$ unless $n \equiv 0 \pmod 4$

The degree of a vertex is the number of edges incident with that vertex. Above left, a graphical representation of the affine plane of order 3 is already properly coloured by its degrees. In the middle, the Petersen graph is clearly not, since it is 3-regular (every vertex has degree 3) but doubling five suitably chosen edges shows that it is 2-degree colourable. And, right, doubling edges in the n-cycle is not sufficient, unless n is a multiple of 4, but degree colourability is 3: tripling edges is sufficient. It might appear that we will find examples requiring ever higher edge multiplicities, but the theorem says not: there is a constant K which bounds the required multiplicities for all possible graphs; and the 1-2-3-Conjectures says $K = 3$. This conjecture was resolved in 2024 by Ralph Keusch.

Construction notes: 2002: Michal Karoński, Thomasz Łuczak and Andrew Thomason conjecture that K exists and equals 3. Prove it for 3-colourable graphs. The conjecture acquires the name "1-2-3 Conjecture".

2004: Louigi Addario-Berry, Ketan Dalal, Colin McDiarmid, Bruce Reed and Thomason: K exists and $K=30$ is sufficient.

2004: Addario-Berry, Dalal and Reed: reduce K to 16. (In 2008: they show $k \leq 2$ for 'almost all' graphs).

2008: Tao Wang and Qinglin Yu reduce K to 13.

2009: Maciej Kalkowski, Michał Karoński and Florian Pfender reduce K to 5.

2024: Ralph Keusch resolves the conjecture by showing that K = 3.

Web link: arxiv.org/abs/1211.5122

Further reading: *Graph Theory* by Reinhard Diestel, Springer, 2017.