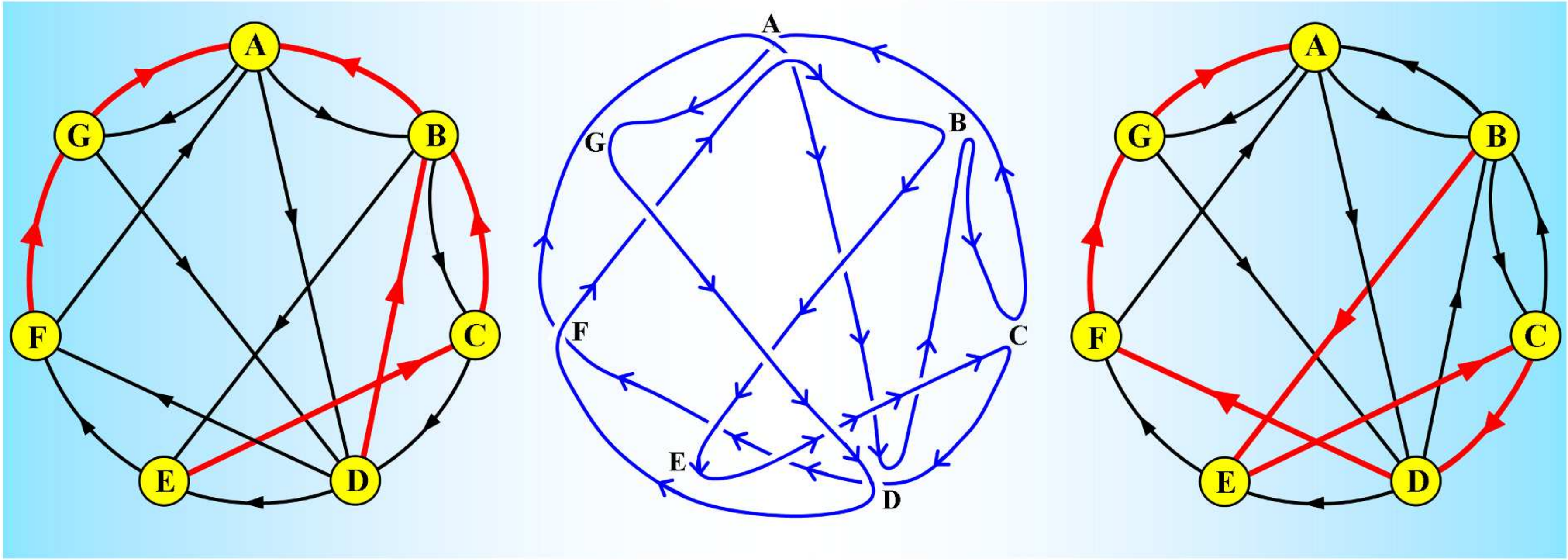
THEOREM OF THE DAY

The BEST Theorem *Let $G = (V, E)$ be a directed graph in which, for each vertex v in V, the indegree and outdegree have the same value, $d(v)$, say. Then G has a directed Euler tour: a closed walk which passes each edge exactly once; let $\varepsilon(G)$ denote the number of such tours. Then, for any fixed vertex x,*

$$\varepsilon(G) = t_x \prod_{v \in V} (d(v) - 1)!$$

where t_x denotes the number of those spanning trees of G in which every vertex has a directed path to x.



A directed graph is shown here with (on the left and right) two spanning trees directed towards vertex A. Choose an edge from A, and from each vertex successively reached, with the rule that tree edges are the last exit chosen from any vertex. The exits from vertex v may be chosen in $(d(v) - 1)!$ orderings; in each case a different Euler tour results. Starting with edge $A \to G$ and using the left-hand tree, the tour shown above centre is one outcome. This tree allows tours to order their exits from A in $d(A)! = 6$ ways but each resulting tour, if traversed commencing with a different initial edge from A, will be found to be counted again by some different tree. Thus, if we follow the tour in the centre starting along edge AD, we will find that the final exit edges are chosen from the right-hand tree. Starting with edge AB will identify a third tree. Thus vertex A contributes a factor of $d(A)!/d(A) = (d(A) - 1)!$ to the product formula, so that it contributes in just the same way as the other vertices albeit for a different reason. By the way, the centre image is reminiscent of a knot diagram and there is indeed a fascinating connection to explore!

The special case of this theorem in which $d(v) = 2$ for every vertex was proved in 1941 by Cedric **S**mith and Bill **T**utte. Nicholaas de **B**ruijn and Tatyana van Aardenne-**E**hrenfest provided the first two initials for the theorem's nickname in 1951, proving the general case and making explicit the link to spanning trees.

Web link: www.cdam.lse.ac.uk/Reports/Files/cdam-2004-12.pdf; see concretenonsense.wordpress.com/2009/08/20/) for a nice account of the connection to knot theory.

Further reading: *A Course in Enumeration* by Martin Aigner, Springer, 2007, Chapter 9.